

## A Century of Surprises

### Chapter 11

Lewinter &amp; Widulski

The Saga of Mathematics

1

## The 19<sup>th</sup> Century

- Critical examination of Euclidean geometry.
- Especially the **parallel postulate** which Euclid took for granted.
- Euclidean versus Non-Euclidean geometry.

Lewinter &amp; Widulski

The Saga of Mathematics

2

## The Parallel Postulate

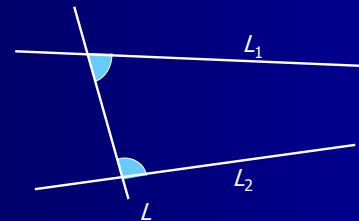
- If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Lewinter &amp; Widulski

The Saga of Mathematics

3

## The Parallel Postulate



Lewinter &amp; Widulski

The Saga of Mathematics

4

## The Parallel Postulate

- The most convincing evidence of Euclid's mathematical genius.
- Euclid had no proof.
- In fact, no proof is possible, but he couldn't go further without this statement.
- Many have tried to prove it but failed.

Lewinter &amp; Widulski

The Saga of Mathematics

5

## Alternatives

- **Poseidonius** (c. 135-51 BC): Two parallel lines are equidistant.
- **Proclus** (c. 500 AD): If a line intersects one of two parallel lines, then it also intersects the other.
- **Saccheri** (c. 1700): The sum of the interior angles of a triangle is two right angles.

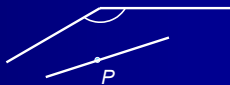
Lewinter &amp; Widulski

The Saga of Mathematics

6

### Alternatives

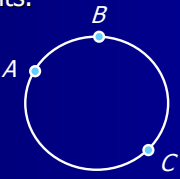
- **Legendre (1752-1833):** A line through a point in the interior of an angle other than a straight angle intersects at least one of the arms of the angle.



Lewinter & Widulski The Saga of Mathematics 7

### Alternatives

- **Farkas Bolyai (1775-1856):** There is a circle through every set of three non-collinear points.



Lewinter & Widulski The Saga of Mathematics 8

### Playfair's Axiom

- *Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.*
  - The Axiom is not Playfair's own invention. He proposed it about 200 years ago, but Proclus stated it some 1300 years earlier.
  - Often substituted for the fifth postulate because it is easier to remember.

Lewinter & Widulski The Saga of Mathematics 9

### Non-Euclidean Geometry

- In the beginning of the last century, some mathematicians began to think along more radical lines.
- Suppose the 5<sup>th</sup> Postulate is not true!
- *"Through a given point in a plane, two lines, parallel to a given straight line, can be drawn."*
- This would change proposition in Euclid.

Lewinter & Widulski The Saga of Mathematics 10

### Non-Euclidean Geometry

- For example: The sum of the angles in a triangle is *less* than two right angles!
- This of course led to Non-Euclidean geometry whose discovery was led by Gauss (1777-1855), Lobachevskii (1792-1856) and Jonas Bolyai (1802-1850).
- Later by Beltrami (1835-1900), Hilbert (1862-1943) and Klein (1849-1945).
- Fits Einstein's *Theory of Relativity*.

Lewinter & Widulski The Saga of Mathematics 11

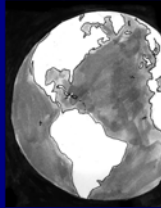
### Non-Euclidean Geometry

- After failing to prove the parallel postulate, mathematicians wondered if there was a consistent "alternative" geometry in which the parallel postulate failed.
- To their amazement, they found two!
- The secret was to look at curved surfaces.
  - The plane is flat – it has no curvature or, as mathematicians say, its curvature is 0.

Lewinter & Widulski The Saga of Mathematics 12

## Geometry on a Sphere

- To do geometry, we need a concept analogous to the straight lines of plane geometry.
- What do straight lines in the plane do?



Lewinter &amp; Widulski

The Saga of Mathematics

13

## Plane Straight Lines

- Firstly, the line segment  $PQ$  yields the shortest distance between points  $P$  and  $Q$ .
- Secondly, a bicyclist traveling from  $P$  to  $Q$  in a straight line will not have to turn his handlebars to the right or left.
  - His motto will be “straight ahead.”

Lewinter &amp; Widulski

The Saga of Mathematics

14

## Geometry on a Sphere

- Similarly, a motorcyclist driving along the equator between two points will be traveling the shortest distance between them and will appear to be traveling straight ahead, even though the equator is curved.
- Like his planar counterpart on the bicycle, our motorcyclist will not have to turn his handlebars to the left or right.
- The same would hold true if he were to travel along a **meridian**, which is sometimes called a **longitude line**.
  - Longitude lines pass through the North and South Poles.

Lewinter &amp; Widulski

The Saga of Mathematics

15

## Geometry on a Sphere

- Meridians and the equator are the result of intersections of the earth with giant planes passing through the center of the earth.
  - For the equator, the plane is horizontal.
  - For the meridians, the planes are vertical.
- There are infinitely many other planes passing through the center of the earth which determine “great circles” which are neither horizontal nor vertical.

Lewinter &amp; Widulski

The Saga of Mathematics

16

## Geometry on a Sphere

- Given two points, such as New York City and London, the shortest route is not a latitude line but rather an arc of the **great circle** formed by intersecting the earth with a plane passing through New York, London and the center of the earth.
- This plane is unique since three non-collinear points in space determine a plane, in a manner analogous to the way two points in the plane determine a line.

Lewinter &amp; Widulski

The Saga of Mathematics

17

## Geometry on a Sphere

- Geometers call a curve on a surface which yields the shortest distance between any two points on it a **geodesic curve** or a **geodesic**.
  - This enables us to do geometry on curved surfaces.
- Imagine a triangle on the earth with one vertex at the North Pole and two others on the equator at a distance  $1/4$  of the circumference of the earth.

Lewinter &amp; Widulski

The Saga of Mathematics

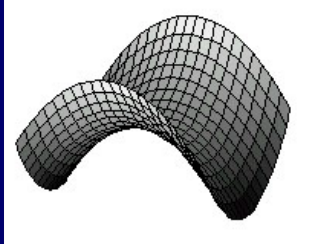
18

## Geometry on a Sphere

- All three angles of this triangle are  $90^\circ$ , so the angle sum is  $270^\circ$ !
- In fact the angle sum of any spherical triangle is larger than  $180^\circ$  and the excess is proportional to its area. Whoa!
- In this geometry, there is no such thing as parallelism.
  - Two great circles must meet in two **antipodal** points – two endpoints of a line passing through the center of the sphere.

Lewinter & Widulski
The Saga of Mathematics
19

## Geometry on a Saddle



Lewinter & Widulski
The Saga of Mathematics
20

## Geometry on a Saddle

- On this kind of surface, parallel geodesics actually diverge!
- They get farther apart, for example, if they go around different sides of its neck.
- The stranger part is that through a point  $P$  not on a given line  $L$  on the surface, there are infinitely many parallel lines.
- Furthermore, angle sums of triangles on these saddle-like surfaces are less than  $180^\circ$ .

Lewinter & Widulski
The Saga of Mathematics
21

## Theory of Relativity

- These two geometries prepared mathematicians and physicists for an even more bizarre geometry required by Albert Einstein (1879 – 1955), whose theory of relativity, in the first half of the 20th century, would shock the world and alter our conception of the physical universe.


Lewinter & Widulski
The Saga of Mathematics
22

## Geodesic Problem

- **The Spider and The Fly:** In a rectangular room  $30' \times 12' \times 12'$  a spider is at the middle of the right wall, one foot below the ceiling. The fly is at the middle of the opposite wall one foot above the floor. The fly is frightened and cannot move. What is the **shortest** distance the spider must crawl in order to capture the fly?

Lewinter & Widulski
The Saga of Mathematics
23

## The Spider and The Fly



Hint: The answer is less than 42...(And the spider must always be in contact with one of the 6 walls).

Lewinter & Widulski
The Saga of Mathematics
24

## Vectors

- A **vector** is best viewed as an arrow.
- It has **magnitude** (length) and **direction**.
- Used to represent velocity or force.
  - Example: A speeding car has a numerical speed, say 60 mph, and a direction, say northeast.
  - The velocity vector of the car can be represented by drawing an arrow of length 60 pointing in the northeast direction.

Lewinter & Widulski      The Saga of Mathematics      25

## The Algebra of Vectors

- Letting bold letters such as **u** and **v** represent vectors, mathematicians and physicists wondered how to do algebra with them, i.e., how to manipulate them in equations as if they were numbers.
- The simplest operation is addition, so what is **u + v**?

Lewinter & Widulski      The Saga of Mathematics      26

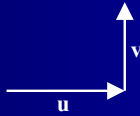
## The Algebra of Vectors

- Picture yourself in a sailboat, and suppose the wind pushes you due east at 8 mph while the current pushes you due north at 6 mph.
- The sum of these vectors should reflect your actual velocity, including both magnitude and direction.

Lewinter & Widulski      The Saga of Mathematics      27

## The Algebra of Vectors

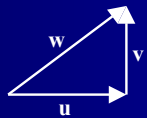
- Since the two velocities (wind and current) act independently, it was realized that the two vectors could be added consecutively, i.e., one after the other.



Lewinter & Widulski      The Saga of Mathematics      28

## The Algebra of Vectors

- The tail of the second vector **v** is placed at the head of the first vector **u**.
- The sum is a vector **w** whose tail is the tail of the first vector and whose head is the head of the second.

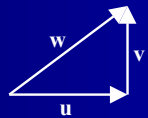


Lewinter & Widulski      The Saga of Mathematics      29

## The Algebra of Vectors

- The magnitude of **w** is easy to find here since the three vectors form a right triangle.
- The Pythagorean Theorem tells us that the length of **w** is

$$\sqrt{8^2 + 6^2} = \sqrt{100} = 10$$



Lewinter & Widulski      The Saga of Mathematics      30

## The Algebra of Vectors

- The speed of the sailboat is 10 mph.
- The boat is not traveling exactly northeast because the angle between vectors  $\mathbf{u}$  and  $\mathbf{w}$  is not  $45^\circ$ .
  - The exact angle may be computed using trigonometry.
- It will be a bit less than  $45^\circ$  since  $\mathbf{v}$  is shorter than  $\mathbf{u}$ .

Lewinter &amp; Widulski

The Saga of Mathematics

31

## The Algebra of Vectors

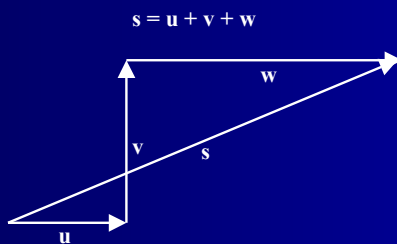
- How do we evaluate sums of three or more vectors? The same way.
- Place them consecutively so that the tail of each vector coincides with the head of the previous one.
- The sum will be a vector whose tail is the tail of the first vector and whose head is the head of the last vector.

Lewinter &amp; Widulski

The Saga of Mathematics

32

## The Sum of Three Vectors



Lewinter &amp; Widulski

The Saga of Mathematics

33

## Vector Notation

- A vector can be described with the use of **components**.
  - Place the vector in  $x, y, z$  space with its tail at the origin.
  - The coordinates of the head are then taken as the components of the vector.
  - We use the notation  $[a, b, c]$  here to distinguish vectors from points, i.e., to distinguish components from coordinates.

Lewinter &amp; Widulski

The Saga of Mathematics

34

## Addition of Vectors

- Mathematicians were delighted to discover that the geometric instructions for addition given above simplify greatly to a mere adding of respective components.
- Thus, if  $\mathbf{u} = [a, b, c]$  and  $\mathbf{v} = [d, e, f]$ , then  $\mathbf{u} + \mathbf{v} = [a + d, b + e, c + f]$ .

Lewinter &amp; Widulski

The Saga of Mathematics

35

## Scalar Multiple

- What would  $2 \times \mathbf{u}$  be?
- It seems that it should correspond to  $\mathbf{u} + \mathbf{u} = [a, b, c] + [a, b, c] = [2a, 2b, 2c]$ .
- This suggests that we have the right to distribute a multiplying number (or scalar) to each component of the vector.

Lewinter &amp; Widulski

The Saga of Mathematics

36

## Summary

- Addition of Vectors:
  - Add the corresponding components.
  - $[a, b, c] + [d, e, f] = [a + d, b + e, c + f]$
- Scalar Multiple:
  - Multiple each component by the scalar.
  - $k \times [a, b, c] = [ka, kb, kc]$

Lewinter & Widulski      The Saga of Mathematics      37

## Example

- Let
  - $\mathbf{u} = [1, -2, 3]$ ,  $\mathbf{v} = [0, 2, 1]$  and  $\mathbf{w} = [-2, 1, -2]$
- Then
  - $\mathbf{u} + \mathbf{v} = [1 + 0, -2 + 2, 3 + 1] = [1, 0, 4]$
  - $\mathbf{u} + \mathbf{w} = [1 + (-2), -2 + 1, 3 + (-2)] = [-1, -1, 1]$
  - $\mathbf{v} + \mathbf{w} = [0 + (-2), 2 + 1, 1 + (-2)] = [-2, 3, -1]$
  - $2\mathbf{u} = [2(1), 2(-2), 2(3)] = [2, -4, 6]$
  - $3\mathbf{w} = 3 \times [-2, 1, -2] = [3(-2), 3(1), 3(-2)] = [-6, 3, -6]$
  - $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [1 - 0, -2 - 2, 3 - 1] = [1, -4, 2]$

Lewinter & Widulski      The Saga of Mathematics      38

## N-dimensional Space

- Mathematicians of the 19th century conjured up an  $n$ -dimensional world, called  $\mathbf{R}^n$ , in which points have  $n$  coordinates and vectors have  $n$  components!
- The above laws carry over quite easily to these  $n$ -dimensional vectors and yield an interesting theory which most find impossible to visualize.
  - $\mathbf{R}^2$  has two axes which are mutually perpendicular (meet at right angles).

Lewinter & Widulski      The Saga of Mathematics      39

## N-dimensional Space

- $\mathbf{R}^3$  has three axes which are mutually perpendicular.
- One adds the  $z$ -axis to the existing set of axes in the plane to get the three dimensional scheme of  $\mathbf{R}^3$ .
- Now what?

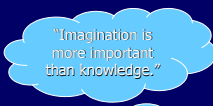
Lewinter & Widulski      The Saga of Mathematics      40

## N-dimensional Space

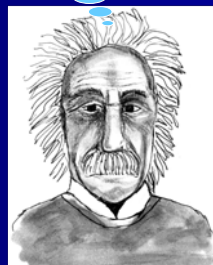
- How do we add a new axis so that it will be perpendicular to the  $x$ ,  $y$ , and  $z$  axis?
- This is where imagination takes over.
- We imagine a new dimension that somehow transcends space and heads off into a fictitious world invisible to non-mathematicians.

Lewinter & Widulski      The Saga of Mathematics      41

## Einstein



- Einstein showed that the universe is four-dimensional.
- Time is the fourth dimension and must be taken into account when computing distance, velocity, force, weight and even length!



Lewinter & Widulski      The Saga of Mathematics      42



## Einstein [1879-1955]

- He posited that large massive objects (like our sun) curve the four-dimensional space around them and cause other objects to follow curved trajectories around them
  - hence the elliptic trajectory of the earth around the sun.
- Einstein correctly predicted that light bends in a gravitational field.

Lewinter &amp; Widulski

The Saga of Mathematics

43

## Einstein [1879-1955]

- This was verified during a solar eclipse at a time when Mercury was on the other side of the sun and normally invisible to us.
- The eclipse, however, rendered it visible and it seemed to be in a slightly different location
  - Precisely accounted for by the bending of light in the gravitational field of the sun.

Lewinter &amp; Widulski

The Saga of Mathematics

44

## Mathematics is the Language of Science

- The 19th century was the time in which electromagnetic phenomena puzzled scientists.
- An electric current in a wire wrapped around a metal rod generated a magnetic field around it.
- On the other hand, a moving magnet generated a current in a wire.
- These phenomena are described by laws using **vector fields**.
  - Spaces in which each point is the tail of a vector whose magnitude and direction varies from point to point.

Lewinter &amp; Widulski

The Saga of Mathematics

45

## The Speed of Light

- The speed of light was measured in two directions:
  - one in the direction of the motion of the earth
  - the other perpendicular to that direction
- The shocking fact was that both speeds were the same.
- It was finally realized that the speed of light seemed independent of the velocity of its source
  - Contradicting the findings of Galileo and Newton.

Lewinter &amp; Widulski

The Saga of Mathematics

46

## Galileo and Newton

- They posited the law of addition of velocities.
  - If a man on a train runs forward at 6 mph and if the train is moving at 60 mph, the speed of the man relative to an observer on the ground is
 
$$60 \text{ mph} + 6 \text{ mph} = 66 \text{ mph}.$$
- Why was light exempt from this law?

Lewinter &amp; Widulski

The Saga of Mathematics

47

## The Speed of Light

- The answer emerged from Einstein's theory that the universe is four dimensional and requires a complicated mathematical scheme of calculation in which the velocity of light, i.e., the speed of propagation of electromagnetic energy, denoted  $c$ , is constant and is in fact the limiting speed of the universe.

Lewinter &amp; Widulski

The Saga of Mathematics

48



$$E = mc^2$$

- This same constant  $c$  plays a role in the conversion of mass into enormous quantities of energy in nuclear reactions, as is predicted by the famous formula:

$$E = mc^2$$

- where  $c \approx 3 \times 10^8$  m/s.

Lewinter &amp; Widulski

The Saga of Mathematics

49

## The Propagation of Light

- The exact nature of light is not fully understood.
- In the 1800s, a physicist Thomas Young showed that light exhibited **wave** characteristics.
- Further experiments by other physicists culminated in James Clerk Maxwell collecting the four fundamental equations that completely describe the behavior of the electromagnetic fields.

Lewinter &amp; Widulski

The Saga of Mathematics

50

## The Propagation of Light

- Maxwell deduced that light was simply a part of the electromagnetic spectrum.
- This seems to firmly establish that light is a **wave**.
- But, in the 1900s, the interaction of light with semiconductor materials, called the photoelectric effect, could not be explained by the electromagnetic-wave theory.

Lewinter &amp; Widulski

The Saga of Mathematics

51

## The Propagation of Light

- The birth of quantum physics successfully explained the photoelectric effect in terms of fundamental particles of energy.
- These particles are called **quanta**.
- Quanta are referred to as **photons** when discussing light energy.

Lewinter &amp; Widulski

The Saga of Mathematics

52

## The Propagation of Light

- Today, when studying light that consists of many photons, as in propagation, that light behaves as a continuum - an electromagnetic wave.
- On the other hand, when studying the interaction of light with semiconductors, as in sources and detectors, the quantum physics approach is taken.
- The wave versus particle dilemma! Oh, no!

Lewinter &amp; Widulski

The Saga of Mathematics

53

## The Sine Function

- The mathematical function that applies to waves is called the **sine** function
  - The behavior of the fluctuating quantity is called **sinusoidal**.
- Originated from the theory of similar triangles first developed in Ancient Greece.
- Two triangles are similar if they have the same angles and their sides are proportional.

Lewinter &amp; Widulski

The Saga of Mathematics

54

### Similar Triangles

- The lengths of sides of the larger triangle are  $k$  times the lengths of the corresponding sides of the smaller.

Lewinter & Widulski The Saga of Mathematics 55

### Similar Triangles

$$\frac{a}{b} = \frac{ka}{kb}, \quad \frac{a}{c} = \frac{ka}{kc}, \quad \text{and} \quad \frac{b}{c} = \frac{kb}{kc}$$

Lewinter & Widulski The Saga of Mathematics 56

### Similar Right Triangles

- Two right triangles are similar if they have the same acute angles (angles less than  $90^\circ$ ).
- Since the two acute angles of a right triangle add up to  $90^\circ$ , all we need to prove similarity is that one of the two angles are the same.
  - For example, if each of two right triangles have a  $30^\circ$  angle, it follows that the other angle must be  $60^\circ$  and the two right angles are similar.
  - Then the ratios of sides are the same in both.

Lewinter & Widulski The Saga of Mathematics 57

### The Sine of an Angle

- The sine of  $\angle A$  of right triangle  $\triangle ABC$ , denoted  $\sin A$ , is the length of the opposite side divided by the length of the hypotenuse, i.e.,  $\sin A = a/c$ .

Lewinter & Widulski The Saga of Mathematics 58

### The Sine of an Angle

- From the point of view of  $\angle A$ , side  $AC$  is called the **adjacent** side and  $BC$  is called the **opposite** side.
- The names are reversed when considering it from the viewpoint of  $\angle B$ .

Lewinter & Widulski The Saga of Mathematics 59

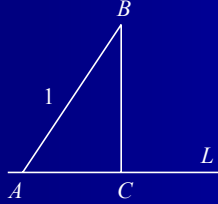
### The Sine of an Angle

- The sine of an **acute** angle is defined as the ratio of the length of the opposite side to the length of the hypotenuse.
 
$$\text{sine} = \text{opposite} / \text{hypotenuse}$$
- Note: It is not necessary to specify the particular right triangle containing the angle.
- The ratio will be the same since all right triangles containing that angle are similar.

Lewinter & Widulski The Saga of Mathematics 60

### The Range of the Sine

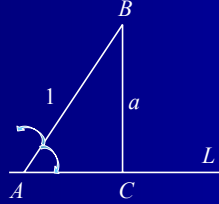
- Imagine a variable right triangle with a hypotenuse  $c$  of length 1
- As shown in the Figure.



Lewinter & Widulski The Saga of Mathematics 61

### The Range of the Sine

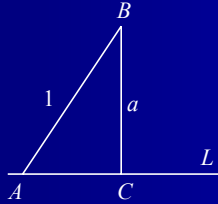
- As  $\angle A$  grows from  $0^\circ$  to  $90^\circ$ , the length of side  $a$  will vary from 0 to 1.
- Thus  $\sin A = a/c = a$  will range from 0 to 1.



Lewinter & Widulski The Saga of Mathematics 62

### The Range of the Sine

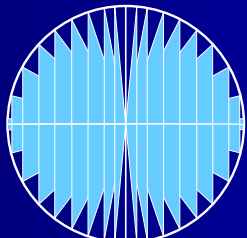
- If we think of  $A$  as a stationary point with a long horizontal line through it, two things should be obvious.



Lewinter & Widulski The Saga of Mathematics 63

### The Range of the Sine

- Firstly,  $\sin A = a/c = a$  or the sine of  $\angle A$  is  $a$  which equals the height of vertex  $B$ .
- Secondly, as  $\angle A$  grows, vertex  $B$  describes an arc of a circle of radius centered at  $A$ .



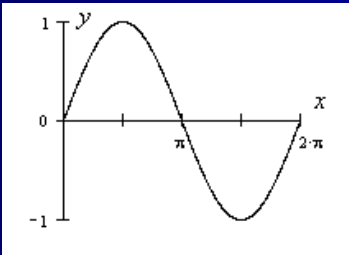
Lewinter & Widulski The Saga of Mathematics 64

### What if the Angle is Obtuse?

- If angle  $A$  is **obtuse**, (between  $90^\circ$  and  $180^\circ$ ), the sine is still defined as the height of vertex  $B$  – even though we no longer have a right triangle.
- If angle  $A$  is larger than  $180^\circ$  (called a **reflex** angle), vertex  $B$  will be under line  $L$  and the sine of angle  $A$  will be a negative number representing the depth of vertex  $B$ .
- As angle  $A$  varies from  $0^\circ$  to  $360^\circ$ , its sine will vary from 0 to 1, back to 0, down to  $-1$ , and finally back up to 0.

Lewinter & Widulski The Saga of Mathematics 65

### The Graph of $y = \sin x$



Lewinter & Widulski The Saga of Mathematics 66

$$y = \sin x$$

- This function can be extended past  $360^\circ$ .
- If angle  $A$  extends to, say,  $370^\circ$ , it will look exactly like  $10^\circ$  and the height of vertex  $B$  will be the same as it was for  $\angle A = 10^\circ$ .
  - From  $360^\circ$  to  $720^\circ$ , it repeats its S-shaped curve.
  - From  $720^\circ$  to  $1080^\circ$ , this same curve will repeat once more, and so on to infinity.
- The sine function is periodic!

Lewinter &amp; Widulski

The Saga of Mathematics

67

$$y = k \sin x$$

- If we graph  $y = 10\sin x$ , we get almost the same graph.
- The new graph will have heights which vary between 10 and  $-10$  instead of between 1 and  $-1$ .
- In the function  $y = k \sin x$ , the height  $k$  is called the **amplitude** of the sine wave.

Lewinter &amp; Widulski

The Saga of Mathematics

68

$$y = \sin (nx)$$

- On the other hand, how would the graph be affected if the function were  $y = \sin (2x)$  or  $y = \sin (3x)$ ?
- In the first case, as  $x$  varies from  $0^\circ$  to  $180^\circ$ , we would get one complete cycle of the sine wave, since  $2x$  would go from  $0^\circ$  to  $360^\circ$ .
- Then the complete cycle would occur again as  $x$  went from  $180^\circ$  to  $360^\circ$ , since  $2x$  would go from  $360^\circ$  to  $720^\circ$ .

Lewinter &amp; Widulski

The Saga of Mathematics

69

$$y = \sin (nx)$$

- In the case of  $y = \sin (3x)$ , as  $x$  varies from  $0^\circ$  to  $120^\circ$ , we would get a complete cycle since  $3x$  would go from  $0^\circ$  to  $360^\circ$ .
- By the time  $x$  gets to  $360^\circ$ , we would have three complete cycles.
- We define the number  $n$  in the equation  $y = \sin (nx)$  to be the **frequency** of the wave because it tells us how many times the complete cycle occurs as  $x$  goes from  $0^\circ$  to  $360^\circ$ .

Lewinter &amp; Widulski

The Saga of Mathematics

70

## AM/FM Radio Waves

- Your radio picks up sinusoidal electromagnetic waves in one of two forms:
  1. Amplitude Modulation (AM) – the amplitude changes while the frequency stays constant.
  2. Frequency Modulation (FM) – the frequency changes while the amplitude stays constant.

Lewinter &amp; Widulski

The Saga of Mathematics

71