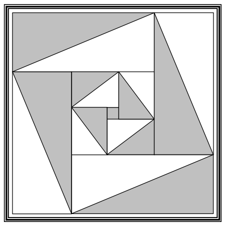
Chapter 13



Some More Math Before You Go

Overview

- The Quadratic Formula
  - The Discriminant
- Multiplication of Binomials – F.O.I.L.
- Factoring
  - Zero factor property
- Graphing Parabolas
  - The Axis of Symmetry, Vertex and Intercepts

Overview

- Simultaneous Equations
- Gabriel Cramer (1704-1752)
  - Cramer's rule
  - Determinants
- Algebraic Fractions
- Equation of a Circle

The Quadratic Formula

- Remember the quadratic formula for solving equations of the form

$$ax^2 + bx + c = 0$$

in which a, b and c represent constants.

The Quadratic Formula

- It is easy to determine these constants from a given quadratic.
- In the equation $2x^2 + 3x – 5 = 0$, we have
  $a = 2$, $b = 3$, and $c = –5$.
- At times, b or c can be zero, as in the equations
  $x^2 – 9 = 0$ and $x^2 – 8x = 0$,
  respectively.

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Usually yields two different solutions in light of the *plus or minus* sign (±).
- The quantity $b^2 – 4ac$ is called the *discriminant*.
- It determines the number and type of solutions.

The Discriminant

- If $b^2 - 4ac > 0$, then there are two real roots.
- If $b^2 - 4ac = 0$, then there is one real (repeated) root, given by $x = -b/2a$.
- If $b^2 - 4ac < 0$, that is, if it is a negative number, it follows that no real number satisfies the quadratic equation.
- This is because the square root of a negative number is not real!

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Imaginary Numbers

- Mathematicians invented the so-called *imaginary number i* to deal with the square roots of negative numbers.

$$i = \sqrt{-1}$$

- If one accepts this bizarre invention, every negative number has a square root, e.g.,

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i$$

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Example

- Let's solve the quadratic equation

$$x^2 - 5x + 6 = 0$$

using the formula.
- Note that $a = 1$, $b = -5$, and $c = 6$.
- The discriminant is

$$(-5)^2 - 4(1)(6) = 25 - 24 = 1$$

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Example (continued)

- It's positive so we will have two solutions.
- They are denoted using subscripts:

$$x_1 = \frac{5+1}{2} = 3 \quad \text{and} \quad x_2 = \frac{5-1}{2} = 2$$

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Problem Solving (Falling Objects)

- In physics, the height H of an object dropped off a building is modeled by the quadratic equation $H = -16T^2 + H_0$ where T is time (in seconds) elapsed since the object was dropped and H_0 is the height (in feet) of the building.
- If the building is 100 feet tall, determine at what time the object hits the ground?

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Multiplication of Binomials

- Example: Multiply $2x + 1$ by $x - 3$
- The work is arranged in columns just as you would do with ordinary numbers.

$$\begin{array}{r} 2x+1 \\ \times \quad x-3 \\ \hline -6x-3 \\ 2x^2+x \quad \\ \hline 2x^2-5x-3 \end{array}$$

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F.O.I.L.

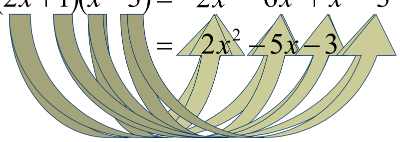
- The first term in the answer, $2x^2$, is the product of the *first* terms of the factors, i.e., $2x$ and x, .
- The last term in the answer, –3, is the product of the *last* terms of the factors, i.e., +1 and –3.
- The middle term, $-5x$, is the result of adding the *outer* and *inner* products of the factors when they are written next to each other as $(2x + 1)(x - 3)$.

F.O.I.L.

- The outer product is $2x \times -3$, or $-6x$, while the inner product is $(+1) \times x$, or $+x$.
- As the middle column of our chart shows, $-6x$ and $+x$ add up to $-5x$.
- You may remember this by its acronym F.O.I.L. which stands for first, outer, inner and last.

F.O.I.L.

$$(2x+1)(x-3) = 2x^2 - 6x + x - 3$$
$$= 2x^2 - 5x - 3$$



First: $2x \times x = 2x^2$ Outer: $2x \times (-3) = -6x$

Inner: $(+1) \times x = +x$ Last: $(+1) \times (-3) = -3$

Factoring

- Let's look at the quadratic equation
  $$x^2 - 5x + 6 = 0$$
- Can this be factored into simple expressions of first *degree*, i.e., expressions involving x to the first power?
- x^2 is just x times x.
- On the other hand, $6 = 6 \times 1 = -6 \times -1 = 3 \times 2 = -3 \times -2$.

Factoring

- How do we decide between the possible answers
  $$(x + 6)(x + 1)$$
  $$(x - 6)(x - 1)$$
  $$(x + 3)(x + 2)$$
  $$(x - 3)(x - 2)$$
- The middle term, $-5x$, is the result of adding the inner and outer products of the last pair of factors!

Factoring

- So the quadratic equation
  $$x^2 - 5x + 6 = 0$$
- Can be written as
  $$(x - 3)(x - 2) = 0$$
- How can the product of two numbers be zero?
- Answer: One (or both) of them must be zero!

Zero Factor Property

- Zero Factor Property:
  If $ab = 0$, then $a = 0$ or $b = 0$.
- To solve the equation for x, we equate each factor to 0.
- If $x - 3 = 0$, x must be 3, while if $x - 2 = 0$, x must be 2.
- This agrees perfectly with the solutions obtained earlier using the quadratic formula.

Graphing Parabolas

- Quadratic expressions often occur in equations that describe a relationship between two variables, such as distance and time in physics, or price and profit in economics.
- The graph of the relationship

$$y = ax^2 + bx + c$$

is a parabola.

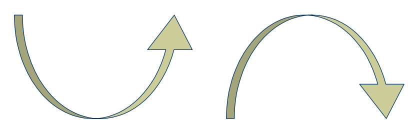
Graphing Parabolas

- Follow these easy steps and you will never fear graphing parabolas ever again.
  1. Determine whether the parabola opens up or down.
  2. Determine the axis of symmetry.
  3. Find the vertex.
  4. Plot a few extra well-chosen points.

Graphing Parabolas

- If a is positive, the parabola opens upward. If a is negative, the parabola is upside down (like the trajectory of a football).

$a > 0$ $a < 0$



The Axis of Symmetry

- Every parabola has an *axis of symmetry* – a vertical line acting like a mirror through which one half of the parabola seems to be the reflection of the other half.
- The equation of the axis of symmetry is

$$x = \frac{-b}{2a}$$

The Vertex

- The *vertex* is a very special point that lays on the parabola.
  - It is the lowest point on the parabola, if the parabola opens up ($a > 0$), or
  - It is the highest point on the parabola, if the parabola opens down ($a < 0$).
- It lays on the axis of symmetry, making its x-coordinate equal to $-b/2a$.

The Vertex

- To get the y-coordinate, insert this x value into the original quadratic expression.
- Finally, pick a few well-chosen x values and find their corresponding y values with the help of the equation and plot them.
- Then connect the dots and you shall have a decent sketch indeed.

The Intercepts

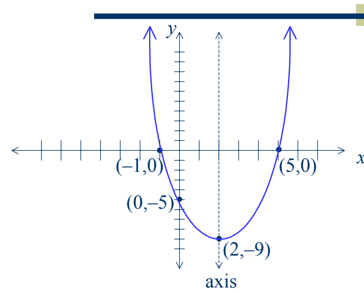
- Sometimes the easiest points to plot are the *intercepts*, i.e., the points where the parabola intersects the x– and y– axes.
  - To find the x-intercepts, insert y=0 into the original quadratic expression and solve for x using the quadratic formula or by factoring.
  - The y-intercept is given by the point (0, c).
- Let's do an example.

Example

- Graph the parabola $y = x^2 - 4x - 5$.

1. Since $a = 1 > 0$, the parabola opens up.
2. The equation of the axis of symmetry is $x = -(-4)/(2\times1) = 2$.
3. Plugging this value into the equation, tells us that the vertex is (2, –9).
4. Solving $x^2 - 4x - 5 = 0$ will give us the x-intercepts of (–1, 0) and (5, 0).

Example (continued)



Linear Equations $ax + by = c$

- Solve the equation $x + y = 10$.
- There are infinitely many pairs of values which satisfy this equation.
  - Including $x = 3$, $y = 7$, and $x = -5$, $y = 15$, and $x = -100$, $y = 110$. Then there are decimal solutions like $x = 2.7$, $y = 7.3$, and so on and so forth.
- The solution set is represented by its graph which is a straight line.

Simultaneous Equations

- Simultaneous equations or a "system of equations" is a collection of equations for which simultaneous solutions are sought.
- Example: $x + y = 10$
  $x - y = 6$
- Solving a system of equations involves finding solutions that satisfy all of the equations.
- The system above has x=8 and y=2 or (8, 2) as a solution.

Simultaneous Equations

- Recall, the solution set for a linear equation in two variables is a straight line.
- Geometrically, we are looking for the intersection of the two straight lines.
- There are three possibilities:
  - There is a unique solution (i.e. the lines intersect in exactly one point)
  - There is no solution (i.e., the lines do not intersect at all)
  - There are infinitely many solutions (i.e., the lines are identical)

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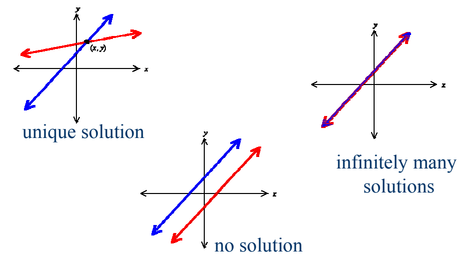
Geometrically



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Simultaneous Equations

- When there is no solution, we say the system is *inconsistent*.
- There are several methods for solving systems of equations including:
  - The Graphing Method
  - The Substitution Method
  - The Method of Addition (or Elimination)
  - Using Determinants

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Simultaneous Equations

- To solve a system by the **graphing method**, graph both equations and determine where the graphs intersect.
- To solve a system by the **substitution method:**
  1. Select an equation and solve for one variable in terms of the other.
  2. Substitute the expression resulting from Step 1 into the other equation to produce an equation in one variable.
  3. Solve the equation produced in Step 2.
  4. Substitute the value for the variable obtained in Step 3 into the expression obtained in Step 1.
  5. Check your solution!

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Simultaneous Equations

- To solve a system by the **addition (or elimination) method**:
1. Multiply either or both equations by nonzero constants to obtain opposite coefficients for one of the variables in the system.
2. Add the equations to produce an equation in one variable and solve this equation.
3. Substitute the value of the variable found in Step 2 into either of the original equations to obtain another equation in one variable. Solve this equation.
4. Check your answer!

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Example

Solve: $\begin{cases} x + y = 10 \\ 5x - y = 14 \end{cases}$

- Since the coefficients of y are opposites, we can add the two equations to obtain $6x = 24$.
- Solving for x gives $x = 4$. We then substitute this into one of the original equations and solve for y which results in $y = 6$.
- So the solution is (4, 6).

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Problem Solving

- A plane with a tailwind flew 5760 miles in 8 hours. On the return trip, against the wind, the plane flew the same distance in 12 hours. What is the speed of the plane in calm air and the speed of the tailwind?
  - Let x = speed of the plane in calm air
    y = speed of the tailwind
- Use the formula: *distance* = *rate* × *time*

Problem Solving

	rate	*time*	*distance*
With the wind	$x + y$	8	$8(x + y)$
Against the wind	$x - y$	12	$12(x - y)$

- This yields the following system:

$$8x + 8y = 5760$$
$$12x - 12y = 5760$$

- Use the method of elimination to solve.

Gabriel Cramer (1704-1752)

- Cramer published articles on a wide range of topics:
  - geometry, philosophy, the aurora borealis, the law, and of course, the history of mathematics.
- He also worked with many famous mathematicians, including Euler, and was held in such high regard that many of them insisted he alone edit their work.

Determinants

- Determinants are mathematical objects which are very useful in the analysis and solution of systems of linear equations.
- A determinant is a function that operates on a square array of numbers.
- The 2×2 determinant is defined as

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = A \times D - B \times C$$

Example

$$\begin{vmatrix} 4 & -2 \\ 3 & 2 \end{vmatrix} = 4 \times 2 - 3 \times (-2)$$
$$= 8 - (-6)$$
$$= 8 + 6$$
$$= 14$$

Cramer's Rule

- Cramer's rule says that the solution of a system such as

$$Ax + By = C$$
$$Dx + Ey = F$$

can be found by calculating the three determinants D_x, D_y and D.
- These three are defined by the following:

Cramer's Rule

$$D_x = \begin{vmatrix} C & B \\ F & E \end{vmatrix}, \quad D_y = \begin{vmatrix} A & C \\ D & F \end{vmatrix},$$

$$\text{and} \quad D = \begin{vmatrix} A & B \\ D & E \end{vmatrix}$$

Cramer's Rule

- The determinant D is called the *determinant of coefficients*, since it contains the coefficients of the variables in the system.
- The *x-determinant* D_x is simply D with the column containing the coefficients of x replaced by the constants from the system, namely, C and F.
- Similarly for the *y-determinant* D_y.

Cramer's Rule

- The solution of the system is

$$x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}$$

provided of course that D is not 0.

Example

- Solve the following system using Cramer's rule:

$$x + y = 4$$
$$3x + 4y = 5$$

- The three determinants D_x, D_y and D are given on the next slide.

Example (continued)

$$D = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 1 = 4 - 3 = 1$$

$$D_x = \begin{vmatrix} 4 & 1 \\ 5 & 4 \end{vmatrix} = 4 \times 4 - 5 \times 1 = 16 - 5 = 11$$

$$D_y = \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} = 1 \times 5 - 3 \times 4 = 5 - 12 = -7$$

Example (continued)

- Since $D \neq 0$, the system has a solution, namely

$$x = \frac{D_x}{D} = \frac{11}{1} = 11$$

and

$$y = \frac{D_y}{D} = \frac{-7}{1} = -7$$

Algebraic Fractions

- In order to add (or subtract) algebraic fractions, you need to find the LCD.
- To find the LCD:
  1. Factor each denominator completely, using exponents to express repeated factors.
  2. Write the product of all the different factors.
  3. For each factor, use the highest power of that factor in any of the denominators.

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To Add (or Subtract) Algebraic Fractions

1. Find the LCD
2. Multiply each expression by
   missing factors
   missing factors
3. Add (or Subtract)
4. Simplify

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Example

Add: $\frac{3}{x-2}+\frac{3}{x+2}=$ $\qquad LCD=(x-2)(x+2)$

$$=\frac{3(x+2)}{(x-2)(x+2)}+\frac{3(x-2)}{(x-2)(x+2)}$$

$$=\frac{3(x+2)+3(x-2)}{(x-2)(x+2)}=\frac{3x+6+3x-6}{(x-2)(x+2)}$$

$$=\frac{6x}{(x-2)(x+2)}$$

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Solving Equations Involving Algebraic Fractions

- Multiply both sides of the equation by the LCD.
- Be sure to use the distributive property when necessary!
- Solve the resulting equation after simplifying both sides.

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Example

Solve: $\frac{x}{x-5}+\frac{5}{x}=\frac{11}{6}$ $\qquad LCD=6x(x-5)$

$$6x(x-5)\left(\frac{x}{x-5}+\frac{5}{x}\right)=6x(x-5)\left(\frac{11}{6}\right)$$

$$6x(x-5)\left(\frac{x}{x-5}\right)+6x(x-5)\left(\frac{5}{x}\right)=6x(x-5)\left(\frac{11}{6}\right)$$

$$6x(x)+6(x-5)(5)=x(x-5)(11)$$

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Example (continued)

$$6x^2+30x-150=11x^2-55x$$

$$0=5x^2-85x+150$$

$$0=x^2-17x+30$$

$$0=(x-2)(x-15)$$

$$x=2 \quad \text{or} \quad x=15$$

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Example (continued)

- Be sure to check your answers and watch out for *extraneous roots*.
- Remember you can <u>not</u> divide by zero, so if an answer results in a denominator having a value of zero that answer is an *extraneous root*.
- Let's look at an example!

Example

Solve: $$\frac{x-3}{x+2} = 3 - \frac{1-2x}{x+2}$$

Problem Solving (Work)

- Neil needs 15 minutes to do the dishes, while Bob can do them in 20 minutes. How long will it take them if they work together?
  - Let x = the time it takes them working together
- Neil can do 1/15 of the work in 1 minute.
- Bob can do 1/20 of the work in 1 minute.
- Together: $\frac{1}{15} + \frac{1}{20} = \frac{1}{x}$

Problem Solving (Uniform Motion)

- Judy drove her empty trailer 300 miles to Saratoga to pick up a load of horses. When the trailer was finally loaded, her average speed was 10 mph less than when the trailer was empty. If the return trip took her 1 hour longer, what was her average speed with the trailer empty?
  - Let x = the average speed
- Use the formula: *distance* = *rate* × *time*

Equation of a Circle

- To find the equation of a circle centered at (a, b) with radius R, we use Pythagorean theorem.
- It follows that $(x - a)^2 + (y - b)^2 = R^2$.