

A Century of Surprises

Chapter 11

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The 19th Century

- Critical examination of Euclidean geometry.
- Especially the **parallel postulate** which Euclid took for granted.
- Euclidean versus Non-Euclidean geometry.

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The Parallel Postulate

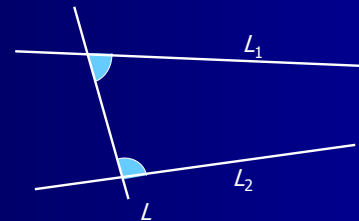
- If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

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The Parallel Postulate



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The Parallel Postulate

- The most convincing evidence of Euclid's mathematical genius.
- Euclid had no proof.
- In fact, no proof is possible, but he couldn't go further without this statement.
- Many have tried to prove it but failed.

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Alternatives

- **Poseidonius** (c. 135-51 BC): Two parallel lines are equidistant.
- **Proclus** (c. 500 AD): If a line intersects one of two parallel lines, then it also intersects the other.
- **Saccheri** (c. 1700): The sum of the interior angles of a triangle is two right angles.

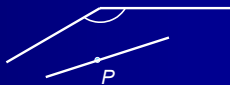
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Alternatives

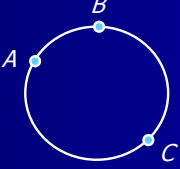
- **Legendre** (1752-1833): A line through a point in the interior of an angle other than a straight angle intersects at least one of the arms of the angle.



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Alternatives

- **Farkas Bolyai** (1775-1856): There is a circle through every set of three non-collinear points.



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Playfair's Axiom

- *Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.*
 - The Axiom is not Playfair's own invention. He proposed it about 200 years ago, but Proclus stated it some 1300 years earlier.
 - Often substituted for the fifth postulate because it is easier to remember.

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Non-Euclidean Geometry

- In the beginning of the last century, some mathematicians began to think along more radical lines.
- Suppose the 5th Postulate is not true!
- *"Through a given point in a plane, two lines, parallel to a given straight line, can be drawn."*
- This would change proposition in Euclid.

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Non-Euclidean Geometry

- For example: The sum of the angles in a triangle is *less* than two right angles!
- This of course led to Non-Euclidean geometry whose discovery was led by Gauss (1777-1855), Lobachevskii (1792-1856) and Jonas Bolyai (1802-1850).
- Later by Beltrami (1835-1900), Hilbert (1862-1943) and Klein (1849-1945).
- Fits Einstein's *Theory of Relativity*.

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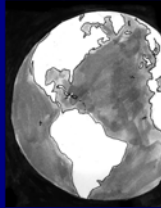
Non-Euclidean Geometry

- After failing to prove the parallel postulate, mathematicians wondered if there was a consistent "alternative" geometry in which the parallel postulate failed.
- To their amazement, they found two!
- The secret was to look at curved surfaces.
 - The plane is flat – it has no curvature or, as mathematicians say, its curvature is 0.

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Geometry on a Sphere

- To do geometry, we need a concept analogous to the straight lines of plane geometry.
- What do straight lines in the plane do?



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Plane Straight Lines

- Firstly, the line segment PQ yields the shortest distance between points P and Q .
- Secondly, a bicyclist traveling from P to Q in a straight line will not have to turn his handlebars to the right or left.
 - His motto will be “straight ahead.”

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Geometry on a Sphere

- Similarly, a motorcyclist driving along the equator between two points will be traveling the shortest distance between them and will appear to be traveling straight ahead, even though the equator is curved.
- Like his planar counterpart on the bicycle, our motorcyclist will not have to turn his handlebars to the left or right.
- The same would hold true if he were to travel along a **meridian**, which is sometimes called a **longitude line**.
 - Longitude lines pass through the North and South Poles.

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Geometry on a Sphere

- Meridians and the equator are the result of intersections of the earth with giant planes passing through the center of the earth.
 - For the equator, the plane is horizontal.
 - For the meridians, the planes are vertical.
- There are infinitely many other planes passing through the center of the earth which determine “great circles” which are neither horizontal nor vertical.

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Geometry on a Sphere

- Given two points, such as New York City and London, the shortest route is not a latitude line but rather an arc of the **great circle** formed by intersecting the earth with a plane passing through New York, London and the center of the earth.
- This plane is unique since three non-collinear points in space determine a plane, in a manner analogous to the way two points in the plane determine a line.

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Geometry on a Sphere

- Geometers call a curve on a surface which yields the shortest distance between any two points on it a **geodesic curve** or a **geodesic**.
 - This enables us to do geometry on curved surfaces.
- Imagine a triangle on the earth with one vertex at the North Pole and two others on the equator at a distance $1/4$ of the circumference of the earth.

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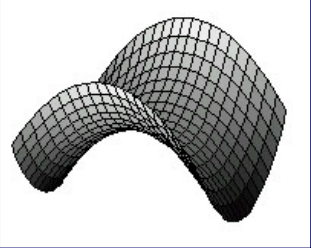
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Geometry on a Sphere

- All three angles of this triangle are 90° , so the angle sum is 270° !
- In fact the angle sum of any spherical triangle is larger than 180° and the excess is proportional to its area. Whoa!
- In this geometry, there is no such thing as parallelism.
 - Two great circles must meet in two **antipodal** points – two endpoints of a line passing through the center of the sphere.

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Geometry on a Saddle



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Geometry on a Saddle

- On this kind of surface, parallel geodesics actually diverge!
- They get farther apart, for example, if they go around different sides of its neck.
- The stranger part is that through a point P not on a given line L on the surface, there are infinitely many parallel lines.
- Furthermore, angle sums of triangles on these saddle-like surfaces are less than 180° .

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Theory of Relativity

- These two geometries prepared mathematicians and physicists for an even more bizarre geometry required by Albert Einstein (1879 – 1955), whose theory of relativity, in the first half of the 20th century, would shock the world and alter our conception of the physical universe.


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Geodesic Problem

- **The Spider and The Fly:** In a rectangular room $30' \times 12' \times 12'$ a spider is at the middle of the right wall, one foot below the ceiling. The fly is at the middle of the opposite wall one foot above the floor. The fly is frightened and cannot move. What is the **shortest** distance the spider must crawl in order to capture the fly?

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The Spider and The Fly



Hint: The answer is less than 42...(And the spider must always be in contact with one of the 6 walls).
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Vectors

- A **vector** is best viewed as an arrow.
- It has **magnitude** (length) and **direction**.
- Used to represent velocity or force.
 - Example: A speeding car has a numerical speed, say 60 mph, and a direction, say northeast.
 - The velocity vector of the car can be represented by drawing an arrow of length 60 pointing in the northeast direction.

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The Algebra of Vectors

- Letting bold letters such as **u** and **v** represent vectors, mathematicians and physicists wondered how to do algebra with them, i.e., how to manipulate them in equations as if they were numbers.
- The simplest operation is addition, so what is **u + v**?

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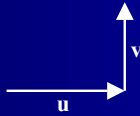
The Algebra of Vectors

- Picture yourself in a sailboat, and suppose the wind pushes you due east at 8 mph while the current pushes you due north at 6 mph.
- The sum of these vectors should reflect your actual velocity, including both magnitude and direction.

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The Algebra of Vectors

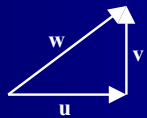
- Since the two velocities (wind and current) act independently, it was realized that the two vectors could be added consecutively, i.e., one after the other.



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The Algebra of Vectors

- The tail of the second vector **v** is placed at the head of the first vector **u**.
- The sum is a vector **w** whose tail is the tail of the first vector and whose head is the head of the second.

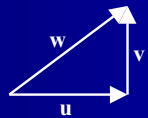


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The Algebra of Vectors

- The magnitude of **w** is easy to find here since the three vectors form a right triangle.
- The Pythagorean Theorem tells us that the length of **w** is

$$\sqrt{8^2 + 6^2} = \sqrt{100} = 10$$



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The Algebra of Vectors

- The speed of the sailboat is 10 mph.
- The boat is not traveling exactly northeast because the angle between vectors \mathbf{u} and \mathbf{w} is not 45° .
 - The exact angle may be computed using trigonometry.
- It will be a bit less than 45° since \mathbf{v} is shorter than \mathbf{u} .

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The Algebra of Vectors

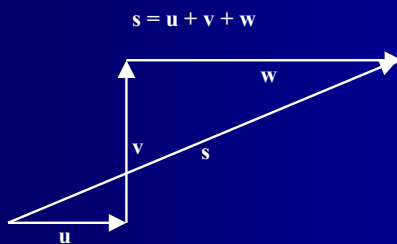
- How do we evaluate sums of three or more vectors? The same way.
- Place them consecutively so that the tail of each vector coincides with the head of the previous one.
- The sum will be a vector whose tail is the tail of the first vector and whose head is the head of the last vector.

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The Sum of Three Vectors



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Vector Notation

- A vector can be described with the use of **components**.
 - Place the vector in x, y, z space with its tail at the origin.
 - The coordinates of the head are then taken as the components of the vector.
 - We use the notation $[a, b, c]$ here to distinguish vectors from points, i.e., to distinguish components from coordinates.

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Addition of Vectors

- Mathematicians were delighted to discover that the geometric instructions for addition given above simplify greatly to a mere adding of respective components.
- Thus, if $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [d, e, f]$, then $\mathbf{u} + \mathbf{v} = [a + d, b + e, c + f]$.

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Scalar Multiple

- What would $2 \times \mathbf{u}$ be?
- It seems that it should correspond to $\mathbf{u} + \mathbf{u} = [a, b, c] + [a, b, c] = [2a, 2b, 2c]$.
- This suggests that we have the right to distribute a multiplying number (or scalar) to each component of the vector.

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Summary

- Addition of Vectors:
 - Add the corresponding components.
 - $[a, b, c] + [d, e, f] = [a + d, b + e, c + f]$
- Scalar Multiple:
 - Multiple each component by the scalar.
 - $k \times [a, b, c] = [ka, kb, kc]$

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Example

- Let
 - $\mathbf{u} = [1, -2, 3]$, $\mathbf{v} = [0, 2, 1]$ and $\mathbf{w} = [-2, 1, -2]$
- Then
 - $\mathbf{u} + \mathbf{v} = [1 + 0, -2 + 2, 3 + 1] = [1, 0, 4]$
 - $\mathbf{u} + \mathbf{w} = [1 + (-2), -2 + 1, 3 + (-2)] = [-1, -1, 1]$
 - $\mathbf{v} + \mathbf{w} = [0 + (-2), 2 + 1, 1 + (-2)] = [-2, 3, -1]$
 - $2\mathbf{u} = [2(1), 2(-2), 2(3)] = [2, -4, 6]$
 - $3\mathbf{w} = 3 \times [-2, 1, -2] = [3(-2), 3(1), 3(-2)] = [-6, 3, -6]$
 - $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [1 - 0, -2 - 2, 3 - 1] = [1, -4, 2]$

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N-dimensional Space

- Mathematicians of the 19th century conjured up an n -dimensional world, called R^n , in which points have n coordinates and vectors have n components!
- The above laws carry over quite easily to these n -dimensional vectors and yield an interesting theory which most find impossible to visualize.
 - R^2 has two axes which are mutually perpendicular (meet at right angles).

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N-dimensional Space

- R^3 has three axes which are mutually perpendicular.
- One adds the z -axis to the existing set of axes in the plane to get the three dimensional scheme of R^3 .
- Now what?

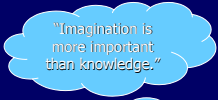
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N-dimensional Space

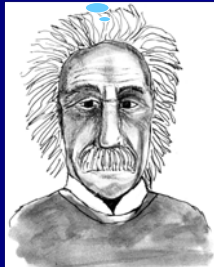
- How do we add a new axis so that it will be perpendicular to the x , y , and z axis?
- This is where imagination takes over.
- We imagine a new dimension that somehow transcends space and heads off into a fictitious world invisible to non-mathematicians.

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Einstein



- Einstein showed that the universe is four-dimensional.
- Time is the fourth dimension and must be taken into account when computing distance, velocity, force, weight and even length!



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Einstein [1879-1955]

- He posited that large massive objects (like our sun) curve the four-dimensional space around them and cause other objects to follow curved trajectories around them
 - hence the elliptic trajectory of the earth around the sun.
- Einstein correctly predicted that light bends in a gravitational field.

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Einstein [1879-1955]

- This was verified during a solar eclipse at a time when Mercury was on the other side of the sun and normally invisible to us.
- The eclipse, however, rendered it visible and it seemed to be in a slightly different location
 - Precisely accounted for by the bending of light in the gravitational field of the sun.

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Mathematics is the Language of Science

- The 19th century was the time in which electromagnetic phenomena puzzled scientists.
- An electric current in a wire wrapped around a metal rod generated a magnetic field around it.
- On the other hand, a moving magnet generated a current in a wire.
- These phenomena are described by laws using **vector fields**.
 - Spaces in which each point is the tail of a vector whose magnitude and direction varies from point to point.

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The Speed of Light

- The speed of light was measured in two directions:
 - one in the direction of the motion of the earth
 - the other perpendicular to that direction
- The shocking fact was that both speeds were the same.
- It was finally realized that the speed of light seemed independent of the velocity of its source
 - Contradicting the findings of Galileo and Newton.

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Galileo and Newton

- They posited the law of addition of velocities.
 - If a man on a train runs forward at 6 mph and if the train is moving at 60 mph, the speed of the man relative to an observer on the ground is

$$60 \text{ mph} + 6 \text{ mph} = 66 \text{ mph}.$$
- Why was light exempt from this law?

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The Speed of Light

- The answer emerged from Einstein's theory that the universe is four dimensional and requires a complicated mathematical scheme of calculation in which the velocity of light, i.e., the speed of propagation of electromagnetic energy, denoted c , is constant and is in fact the limiting speed of the universe.

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$$E = mc^2$$

- This same constant c plays a role in the conversion of mass into enormous quantities of energy in nuclear reactions, as is predicted by the famous formula:

$$E = mc^2$$

- where $c \approx 3 \times 10^8$ m/s.

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The Propagation of Light

- The exact nature of light is not fully understood.
- In the 1800s, a physicist Thomas Young showed that light exhibited **wave** characteristics.
- Further experiments by other physicists culminated in James Clerk Maxwell collecting the four fundamental equations that completely describe the behavior of the electromagnetic fields.

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The Propagation of Light

- Maxwell deduced that light was simply a part of the electromagnetic spectrum.
- This seems to firmly establish that light is a **wave**.
- But, in the 1900s, the interaction of light with semiconductor materials, called the photoelectric effect, could not be explained by the electromagnetic-wave theory.

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The Propagation of Light

- The birth of quantum physics successfully explained the photoelectric effect in terms of fundamental particles of energy.
- These particles are called **quanta**.
- Quanta are referred to as **photons** when discussing light energy.

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The Propagation of Light

- Today, when studying light that consists of many photons, as in propagation, that light behaves as a continuum - an electromagnetic wave.
- On the other hand, when studying the interaction of light with semiconductors, as in sources and detectors, the quantum physics approach is taken.
- The wave versus particle dilemma! Oh, no!

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The Sine Function

- The mathematical function that applies to waves is called the **sine** function
 - The behavior of the fluctuating quantity is called **sinusoidal**.
- Originated from the theory of similar triangles first developed in Ancient Greece.
- Two triangles are similar if they have the same angles and their sides are proportional.

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Similar Triangles

- The lengths of sides of the larger triangle are k times the lengths of the corresponding sides of the smaller.

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Similar Triangles

$$\frac{a}{b} = \frac{ka}{kb}, \quad \frac{a}{c} = \frac{ka}{kc}, \quad \text{and} \quad \frac{b}{c} = \frac{kb}{kc}$$

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Similar Right Triangles

- Two right triangles are similar if they have the same acute angles (angles less than 90°).
- Since the two acute angles of a right triangle add up to 90° , all we need to prove similarity is that one of the two angles are the same.
 - For example, if each of two right triangles have a 30° angle, it follows that the other angle must be 60° and the two right angles are similar.
 - Then the ratios of sides are the same in both.

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The Sine of an Angle

- The sine of $\angle A$ of right triangle $\triangle ABC$, denoted $\sin A$, is the length of the opposite side divided by the length of the hypotenuse, i.e., $\sin A = a/c$.

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The Sine of an Angle

- From the point of view of $\angle A$, side AC is called the **adjacent** side and BC is called the **opposite** side.
- The names are reversed when considering it from the viewpoint of $\angle B$.

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The Sine of an Angle

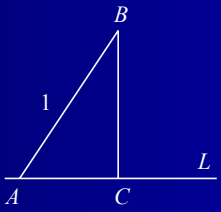
- The sine of an **acute** angle is defined as the ratio of the length of the opposite side to the length of the hypotenuse.

$$\text{sine} = \text{opposite} / \text{hypotenuse}$$
- Note: It is not necessary to specify the particular right triangle containing the angle.
- The ratio will be the same since all right triangles containing that angle are similar.

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The Range of the Sine

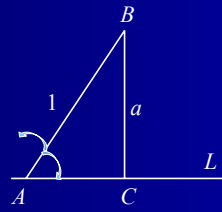
- Imagine a variable right triangle with a hypotenuse c of length 1
- As shown in the Figure.



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The Range of the Sine

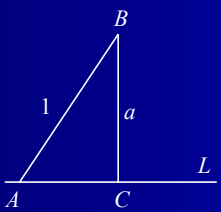
- As $\angle A$ grows from 0° to 90° , the length of side a will vary from 0 to 1.
- Thus $\sin A = a/c = a$ will range from 0 to 1.



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The Range of the Sine

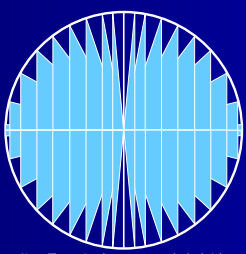
- If we think of A as a stationary point with a long horizontal line through it, two things should be obvious.



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The Range of the Sine

- Firstly, $\sin A = a/c = a$ or the sine of $\angle A$ is a which equals the height of vertex B .
- Secondly, as $\angle A$ grows, vertex B describes an arc of a circle of radius centered at A .



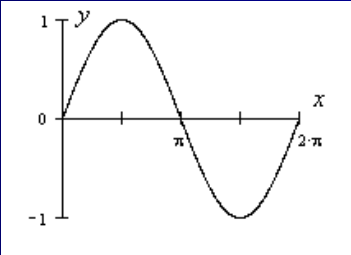
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What if the Angle is Obtuse?

- If angle A is **obtuse**, (between 90° and 180°), the sine is still defined as the height of vertex B – even though we no longer have a right triangle.
- If angle A is larger than 180° (called a **reflex** angle), vertex B will be under line L and the sine of angle A will be a negative number representing the depth of vertex B .
- As angle A varies from 0° to 360° , its sine will vary from 0 to 1, back to 0, down to -1 , and finally back up to 0.

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The Graph of $y = \sin x$



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$$y = \sin x$$

- This function can be extended past 360° .
- If angle A extends to, say, 370° , it will look exactly like 10° and the height of vertex B will be the same as it was for $\angle A = 10^\circ$.
 - From 360° to 720° , it repeats its S -shaped curve.
 - From 720° to 1080° , this same curve will repeat once more, and so on to infinity.
- The sine function is periodic!

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$$y = k \sin x$$

- If we graph $y = 10\sin x$, we get almost the same graph.
- The new graph will have heights which vary between 10 and -10 instead of between 1 and -1 .
- In the function $y = k \sin x$, the height k is called the **amplitude** of the sine wave.

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$$y = \sin (nx)$$

- On the other hand, how would the graph be affected if the function were $y = \sin (2x)$ or $y = \sin (3x)$?
- In the first case, as x varies from 0° to 180° , we would get one complete cycle of the sine wave, since $2x$ would go from 0° to 360° .
- Then the complete cycle would occur again as x went from 180° to 360° , since $2x$ would go from 360° to 720° .

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$$y = \sin (nx)$$

- In the case of $y = \sin (3x)$, as x varies from 0° to 120° , we would get a complete cycle since $3x$ would go from 0° to 360° .
- By the time x gets to 360° , we would have three complete cycles.
- We define the number n in the equation $y = \sin (nx)$ to be the **frequency** of the wave because it tells us how many times the complete cycle occurs as x goes from 0° to 360° .

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AM/FM Radio Waves

- Your radio picks up sinusoidal electromagnetic waves in one of two forms:
 1. Amplitude Modulation (AM) – the amplitude changes while the frequency stays constant.
 2. Frequency Modulation (FM) – the frequency changes while the amplitude stays constant.

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