Mathematics Marches On

Chapter 7

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Background

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The High Middle Ages

- As Europe entered the period known as the High Middle Ages, the church became the universal and unifying institution.
  - It had a monopoly on education
- The music was mainly the Gregorian chant.
  - Monophonic and in Latin
  - Named after Pope Gregory I
- The rise of towns caused economic & social institutions to mature with an era of greater creativity.

The Renaissance

- The Renaissance encouraged freedom of thought.
- For religion, it was a time when many reformers began to question the power of the Roman Catholic church.
- Change began to happen because of the spread of ideas.
- From the French word for “rebirth.”

The Renaissance

- Increased interest in knowledge of all types.
- Education becomes a status symbol, and people are expected to be knowledgeable in many areas of study including **art**, **music**, philosophy, science, and literature.
- Renaissance scholars known as **humanists** returned to the works of ancient writers of Greece and Rome.

The Renaissance

- The recovery of ancient manuscripts showed the humanists how the Greeks and Romans employed **mathematics** to give structure to their art, music, and architecture.
- In architecture, **numerical ratios** were used in building design.
- In art, **geometry** was used in painting.

The Renaissance

- After five-hundred years of Gregorian chants, attempts were made to make music more interesting by dividing the singers into two groups and assigning to each a different melody.
- The idea was simple and brilliant – the hard part was deciding what notes to give the second group.
- The first group sang the original melody.

Mathematics and Music

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Quotes

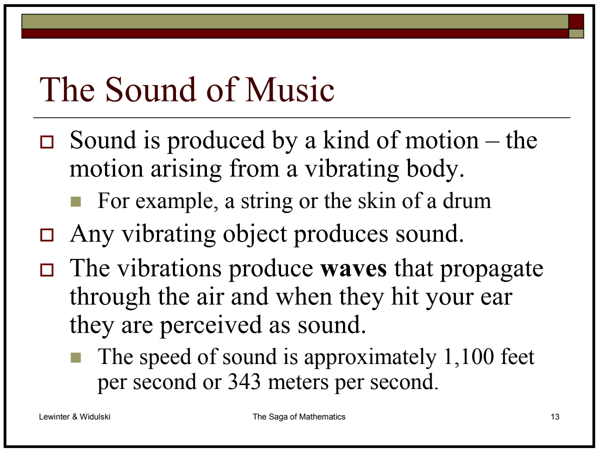
- *"Without music life would be a mistake."* – Friedrich Nietzsche
- *"Music is a secret arithmetical exercise and the person who indulges in it does not realize that he is manipulating numbers."* – Gottfried Liebniz

Math and Music

- Mathematics and music have been link since the days of Pythagoras.
- Since the Middle Ages, music theorists had been studying **proportions**, a subject that Pythagoras had written about when discussing music.
- The theorists explained how to make different pitches (sounds) on stringed instruments by lengthening or shortening the strings by different proportions.

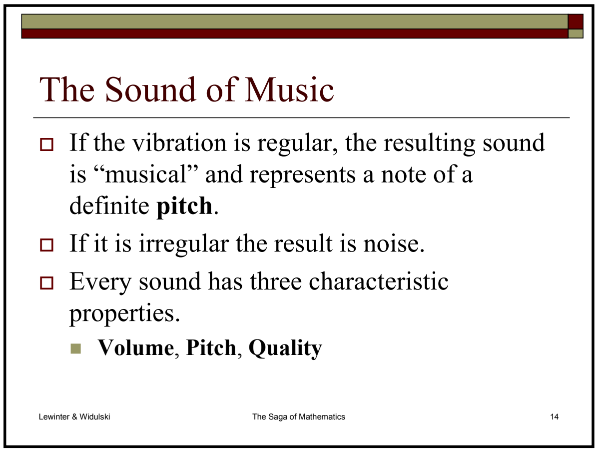
Math and Music

- For example, if a musician were to divide a string in half (the proportion of 2:1), he would create a new tone that is an **octave** above the original tone.
- Renaissance musicians carried on this idea in their own music.



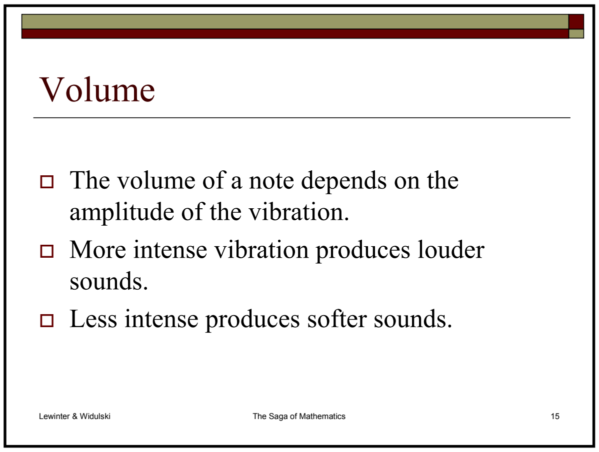
The Sound of Music

- Sound is produced by a kind of motion – the motion arising from a vibrating body.
  - For example, a string or the skin of a drum
- Any vibrating object produces sound.
- The vibrations produce **waves** that propagate through the air and when they hit your ear they are perceived as sound.
  - The speed of sound is approximately 1,100 feet per second or 343 meters per second.



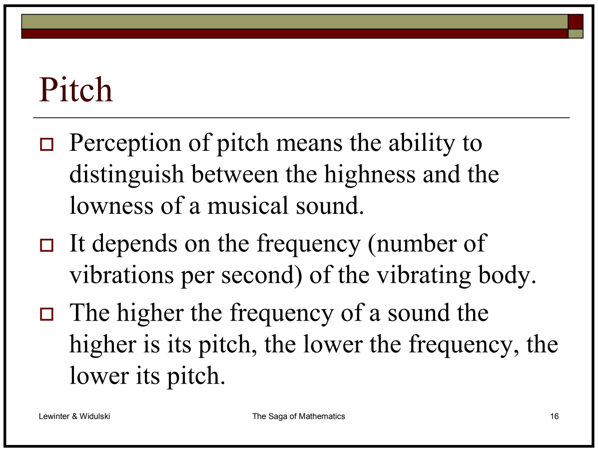
The Sound of Music

- If the vibration is regular, the resulting sound is “musical” and represents a note of a definite **pitch**.
- If it is irregular the result is noise.
- Every sound has three characteristic properties.
  - **Volume**, **Pitch**, **Quality**



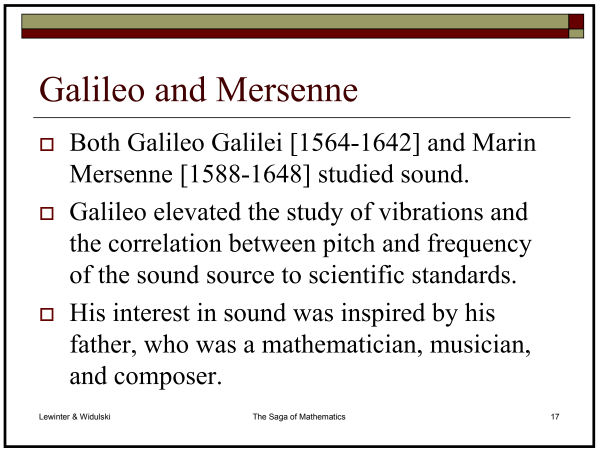
Volume

- The volume of a note depends on the amplitude of the vibration.
- More intense vibration produces louder sounds.
- Less intense produces softer sounds.



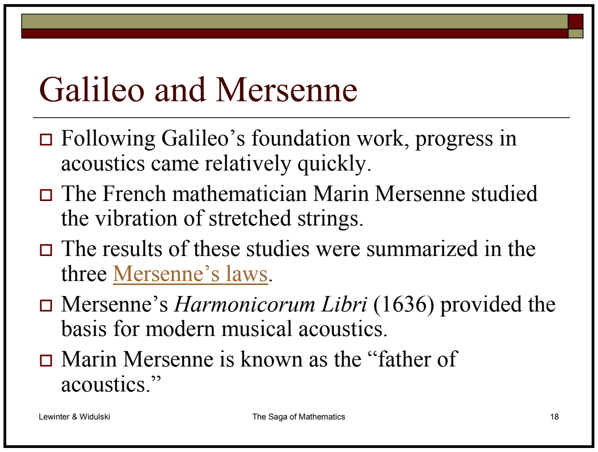
Pitch

- Perception of pitch means the ability to distinguish between the highness and the lowness of a musical sound.
- It depends on the frequency (number of vibrations per second) of the vibrating body.
- The higher the frequency of a sound the higher is its pitch, the lower the frequency, the lower its pitch.



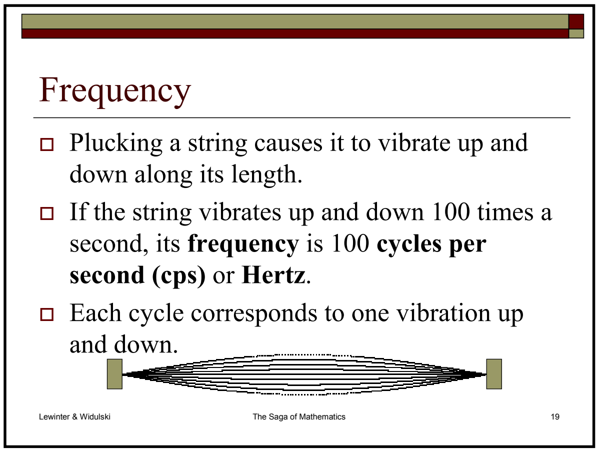
Galileo and Mersenne

- Both Galileo Galilei [1564-1642] and Marin Mersenne [1588-1648] studied sound.
- Galileo elevated the study of vibrations and the correlation between pitch and frequency of the sound source to scientific standards.
- His interest in sound was inspired by his father, who was a mathematician, musician, and composer.



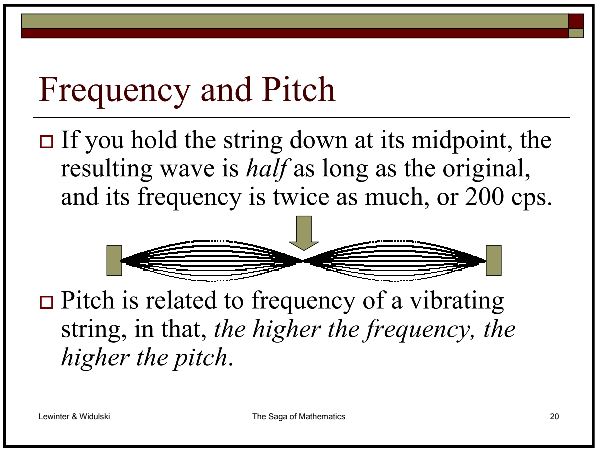
Galileo and Mersenne

- Following Galileo’s foundation work, progress in acoustics came relatively quickly.
- The French mathematician Marin Mersenne studied the vibration of stretched strings.
- The results of these studies were summarized in the three Mersenne’s laws.
- Mersenne’s *Harmonicorum Libri* (1636) provided the basis for modern musical acoustics.
- Marin Mersenne is known as the “father of acoustics.”



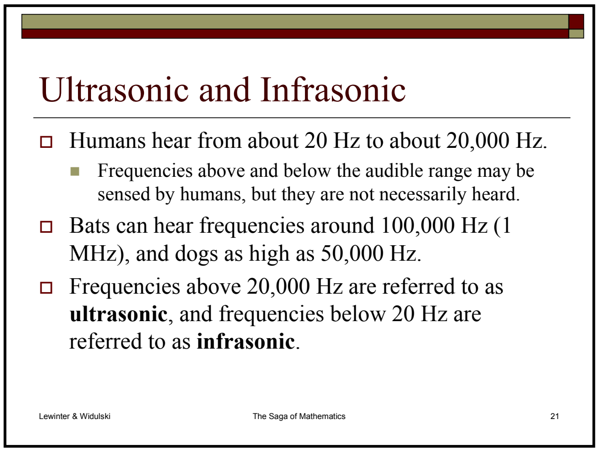
Frequency

- Plucking a string causes it to vibrate up and down along its length.
- If the string vibrates up and down 100 times a second, its **frequency** is 100 **cycles per second (cps)** or **Hertz**.
- Each cycle corresponds to one vibration up and down.



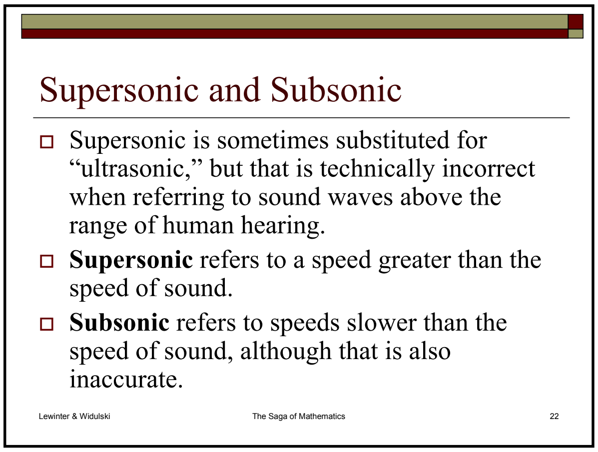
Frequency and Pitch

- If you hold the string down at its midpoint, the resulting wave is *half* as long as the original, and its frequency is twice as much, or 200 cps.
- Pitch is related to frequency of a vibrating string, in that, *the higher the frequency, the higher the pitch*.



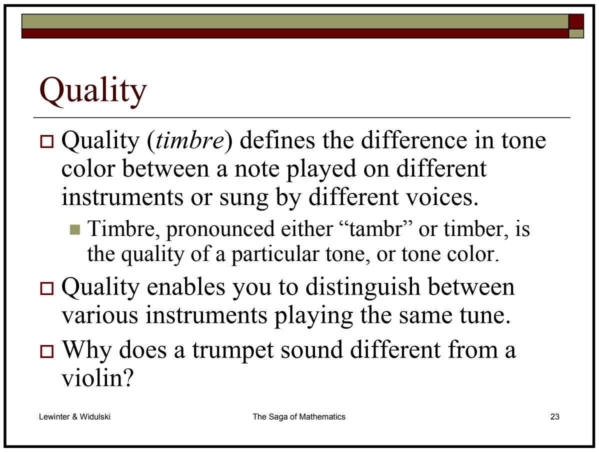
Ultrasonic and Infrasonic

- Humans hear from about 20 Hz to about 20,000 Hz.
  - Frequencies above and below the audible range may be sensed by humans, but they are not necessarily heard.
- Bats can hear frequencies around 100,000 Hz (1 MHz), and dogs as high as 50,000 Hz.
- Frequencies above 20,000 Hz are referred to as **ultrasonic**, and frequencies below 20 Hz are referred to as **infrasonic**.



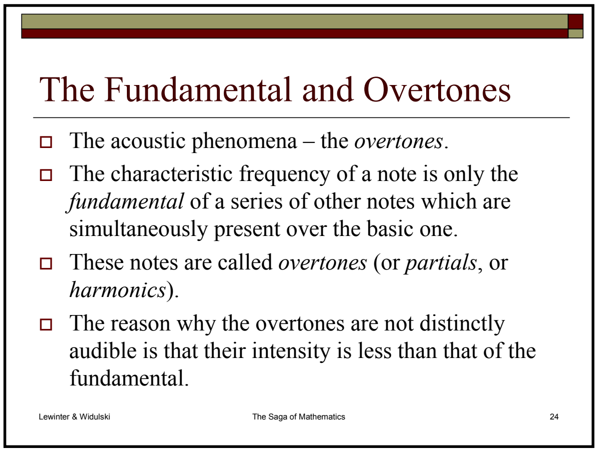
Supersonic and Subsonic

- Supersonic is sometimes substituted for "ultrasonic," but that is technically incorrect when referring to sound waves above the range of human hearing.
- **Supersonic** refers to a speed greater than the speed of sound.
- **Subsonic** refers to speeds slower than the speed of sound, although that is also inaccurate.



Quality

- Quality (*timbre*) defines the difference in tone color between a note played on different instruments or sung by different voices.
  - Timbre, pronounced either "tambr" or timber, is the quality of a particular tone, or tone color.
- Quality enables you to distinguish between various instruments playing the same tune.
- Why does a trumpet sound different from a violin?



The Fundamental and Overtones

- The acoustic phenomena – the *overtones*.
- The characteristic frequency of a note is only the *fundamental* of a series of other notes which are simultaneously present over the basic one.
- These notes are called *overtones* (or *partials*, or *harmonics*).
- The reason why the overtones are not distinctly audible is that their intensity is less than that of the fundamental.

Fundamental Frequency

- The initial vibration of a sound is called the **fundamental**, or **fundamental frequency**.
- In a purely Physics-based sense, the fundamental is the lowest pitch of a sound, and in most real-world cases this model holds true.
- A note played on a string has a fundamental frequency which is its lowest natural frequency.
- Additionally, the fundamental frequency is the strongest pitch we hear.

The Overtones

- The note also has *overtones* at consecutive integer multiples of its fundamental frequency.
- Plucking a string excites a number of tones not just the fundamental.
- They determine the quality of a note and they give brilliance to the tone.
- What makes us able to distinguish between an oboe and a horn is the varying intensity of the overtones over the actual notes which they play.

The Overtones (Harmonics)

- Music would be boring if all sounds were comprised of just their fundamental frequency – you would not be able to tell the difference between a violin playing an “A” at 440 Hz and a flute playing the same note?
- Luckily, most sounds are a combination of a fundamental pitch and various multiples of the fundamental, known as **overtones**, or **harmonics**.
- When overtones are added to the fundamental frequency, the character or quality of the sound is changed; the character of the sound is called **timbre**.

Harmonics and Overtones

- The term **harmonic** has a precise meaning – that of an integer (whole number) multiple of the fundamental frequency of a vibrating object.
- The term **overtone** is used to refer to any resonant frequency above the fundamental frequency.
- Many of the instruments of the orchestra, those utilizing strings or air columns, produce the fundamental frequency and harmonics.

Example

- An instrument playing a note at a fundamental of 200 Hz will have a second harmonic at 400 Hz, a third harmonic at 600 Hz, a fourth harmonic at 800 Hz, ad nauseam.
- What would the first six harmonics be for a fundamental of 440 Hz?

Harmonic Series

- Harmonic Series – a series of tones consisting of a fundamental tone and the overtones produced by it.
- It is the amplitude and placement of harmonics and partials which give different instruments different timbre (despite not usually being detected separately by the untrained human ear).

Harmonic Series

- Given a fundamental of C’, the first 6 harmonics are:
  - 1st C’ – the fundamental
  - 2nd C – the first octave (8th) above
  - 3rd G – the twelve (12th) above
  - 4th c – the second octave (15th) above
  - 5th e – the 17th above
  - 6th g – the 19th above

Psychoacoustics

- The study of **psychoacoustics** teaches us that even-numbered harmonics tend to make sounds “soft” and “warm,” while odd-numbered harmonics make sounds “bright” and “metallic.”
- Lower-order harmonics control the basic timbre of the sound, and higher-order harmonics control the harshness of the sound.

The Octave

- Again, a one-**octave** separation occurs when the higher frequency is twice the lower frequency – the octave ratio is thus 2:1.
- A note’s first overtone is one octave higher than its fundamental frequency.
- An **octave** denotes the difference between any two frequencies where the ratio between them is 2:1.
  - Therefore, an octave separates the fundamental from the second harmonic as above: 400 Hz:200 Hz.

The Octave

- Note that even though, as frequency increases, the linear distance between frequencies becomes greater, the ratio of 2:1 is still the same: an octave still separates 4000 Hz from 2000 Hz.
- In the musical world, two notes separated by an octave are said to be “in tune.”
- An “A” on a violin at 440 Hz is an octave below the “A” at 880 Hz.

Other Ratios

- The most consonant sounds are those of the fundamental, the fifth and the fourth.
- Remember, the Pythagoreans found beauty in the ratios 1:2:3:4.

Ratio	**Name**
1:1	Unison
1:2	Octave
1:3	Twelfth
2:3	Fifth
3:4	Fourth
4:5	Major Third
3:5	Major Sixth

Standard Pitch

- Musicians tune their instruments to a note which has 440 cycles per second.
- This is the accepted number of vibration for the note *a* above middle *c*.
- In 1939 at an international conference most of the Western nations accepted this note as the standard pitch.
  - A – 440Hz

Intonation

- Good intonation means being in tune (pitching the note accurately).
- If two notes have the same frequency, we know that they have the same pitch, and so they are in unison.
- But if one of these is played slightly out of tune, the result is that one produces shorter wave and these waves collide with each other, producing a pulsating effect.

Resonance

- Certain pitches can cause some nearby object to resound sympathetically.
  - Opera singer shattering a glass.
- When two vibrating sources are at the same pitch, and one is set into vibration, the untouched one will take the vibration sympathetically from the other.

Resonance

- When we sing it is not our vocals cords alone which produce sound, but the sympathetic vibrations set up in the cavities of our heads.
- It is the belly of a guitar which actually produces the tone, by vibrating sympathetically with the string.

Resonant Frequencies and Bridges

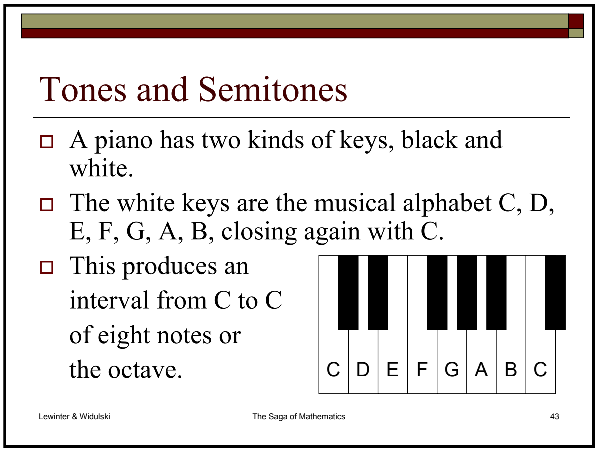
- Bridges have a “natural frequency.”
- When the wind blows or people cross the bridge at a rhythm that matches this frequency, the force can cause the bridge to vibrate.
- This phenomenon is called **resonance**, and the frequency is called **resonant frequency**.

Resonant Frequencies and Bridges

- Soldiers are taught to march across a bridge out-of-step, so they won’t create vibrations that tap into the bridge’s resonant frequency.
- In extreme cases, the vibrations can cause a bridge to collapse, as happened when the driving force of the wind caused the collapse of the Tacoma Narrows Bridge in Washington State in 1940.

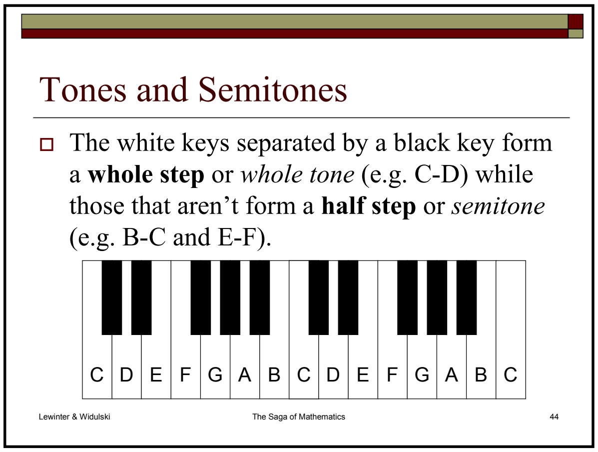
Music Theory

- Musical Notation
- Rhythm
- Tempo and Dynamics
- Tones and Semitones
- Scales - Sharps, Flats, and Naturals
- Tonality
- Intervals



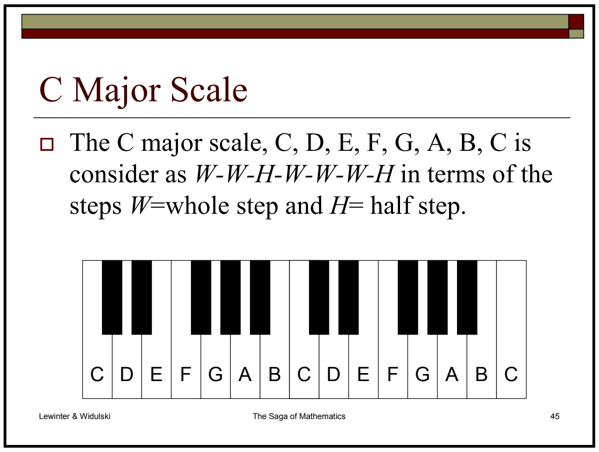
Tones and Semitones

- A piano has two kinds of keys, black and white.
- The white keys are the musical alphabet C, D, E, F, G, A, B, closing again with C.
- This produces an interval from C to C of eight notes or the octave.



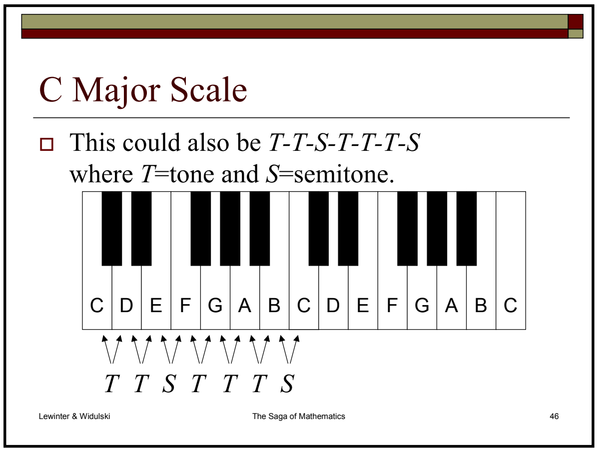
Tones and Semitones

- The white keys separated by a black key form a **whole step** or *whole tone* (e.g. C-D) while those that aren't form a **half step** or *semitone* (e.g. B-C and E-F).



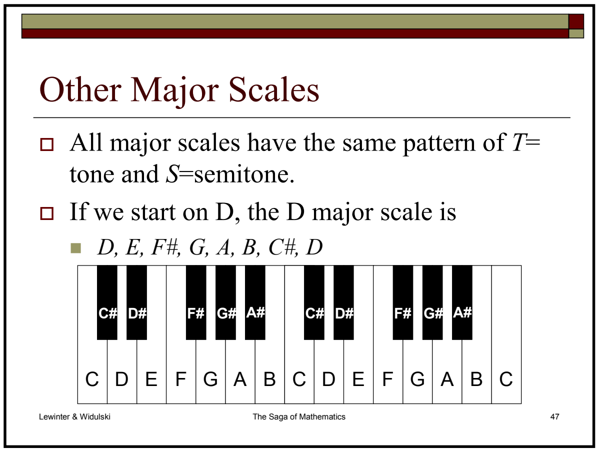
C Major Scale

- The C major scale, C, D, E, F, G, A, B, C is consider as *W-W-H-W-W-W-H* in terms of the steps *W*=whole step and *H*= half step.



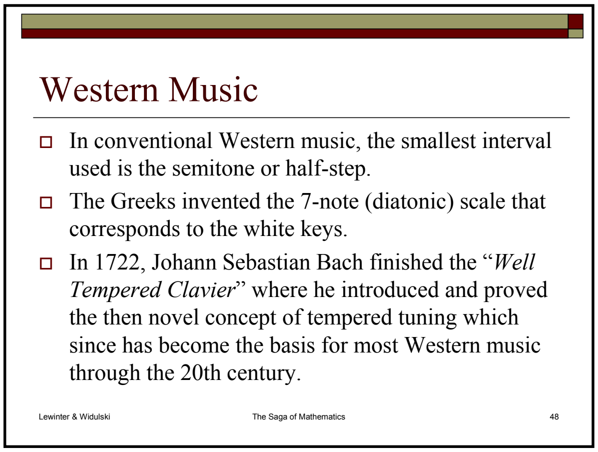
C Major Scale

- This could also be *T-T-S-T-T-T-S* where *T*=tone and *S*=semitone.



Other Major Scales

- All major scales have the same pattern of *T*= tone and *S*=semitone.
- If we start on D, the D major scale is
  - *D, E, F#, G, A, B, C#, D*



Western Music

- In conventional Western music, the smallest interval used is the semitone or half-step.
- The Greeks invented the 7-note (diatonic) scale that corresponds to the white keys.
- In 1722, Johann Sebastian Bach finished the "*Well Tempered Clavier*" where he introduced and proved the then novel concept of tempered tuning which since has become the basis for most Western music through the 20th century.

12-Tone Scale

- On the 12-tone scale, the frequency separating each note is the half-step.

  C→C#→D→D#→E→F→F#→G→G#→A→A#→B→C
- In each half-step, the frequency increases by some multiplicative factor say *f*.
- That is, the frequency of the note C# is the frequency of C times the factor *f*.

  $C \xrightarrow[f]{} C\# \xrightarrow[f]{} D \xrightarrow[f]{} D\# \xrightarrow[f]{} E \xrightarrow[f]{} F \xrightarrow[f]{} F\# \xrightarrow[f]{} G \xrightarrow[f]{} G\# \xrightarrow[f]{} A \xrightarrow[f]{} A\# \xrightarrow[f]{} B \xrightarrow[f]{} C$

12-Tone Scale

- Since an octave corresponds to doubling the frequency, multiplying these 12 factors *f* together should gives 2.
- In other words, $f^{12} = 2$.
- Thus, *f* must be the *twelve root* of two, or

$$f = \sqrt[12]{2} \cong 1.05946$$

12-Tone Scale

- Using this you can calculate every note of the 12-tone scale.
- Starting with middle C whose frequency is 260 cps, multiply by *f* to get the frequency of C#.
- Multiplying again will give the frequency of D.
- Continue until all notes are calculated.

Math and Music Web sites

- Sound Waves and Music at The Physics Classroom.
- Teaching Math with Music at Southwest Educational Development Laboratory.
- The Problem of Temperament
- Time Signatures
- Polyrhythms

Mathematics and Art

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Painting in the Middle Ages

- During the Middle Ages, European artists painted in a way that emphasized religious images and symbolism rather than realism.
- Most paintings depicted scenes holy figures and people important in the Christian religion.
- Even the most talented painters of the Middle Ages paid little attention to making humans and animals look lifelike, creating natural looking landscapes, or creating a sense of depth and space in their paintings.

Painting in the Renaissance

- European artists began to study the model of nature more closely and began to paint with the goal of greater realism.
- They learned to create lifelike people and animals and they became skilled at creating the illusion of depth and distance on walls and canvases by using the techniques of linear perspective.

Perspective

- **Perspective** is a system used by artists, designers, and engineers to represent three-dimensional objects on a two-dimensional surface.
- An artist uses perspective in order to represent nature or objects in the most effective way possible.
- It evolved from "*Costruzione Legittima*" invented sometime in the fifteenth century, most likely by **Fillipo Brunelleschi**.
- **Leon Battista Alberti** and **Piero della Francesca** improved upon Brunelleschi's theories.

Perspective

- The main idea for constructing a proper perspective is the idea of "vanishing points."
- The "principal vanishing point" deals with lines that are parallel to each other and moving away from the artist.
- In one point perspective, the horizon line exists where the viewer's line of sight is.
- Also, in one point perspective, all parallel lines which are perpendicular to the horizon line will converge at a point on the horizon line called the vanishing point.

The Horizon Line

- The horizon line exists wherever your line of sight is.
- It always falls at eye level regardless of where you're looking.
- For instance, if you are looking down, your eye level remains at the height of your eyes, not down where you are looking.

Vanishing Point

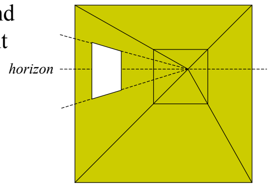
- The point to which all lines which are parallel to the viewer recede.
- Think of the last time you were looking down a long stretch of straight highway.
- The edges of that highway appear to move at an angle upward until they meet the horizon.
- In one point perspective all verticals and horizontals stay the same and only lines that are moving away from or toward the viewer seem to recede on the horizon at the vanishing point.

Convergence Lines

- Lines that converge at the vanishing point.
- These are any lines that are moving away from the viewer at an angle parallel to the direction that the viewer is looking.
- In the case of the highway mentioned above these lines would be the edges of the highway as they move away from you forward into the distance.
- They are also called orthogonals.

Perspective

- To draw in perspective, draw a horizon line and draw a vanishing point anywhere on the horizon.
- Lines which are parallel in real life are drawn to intersect at the vanishing point.

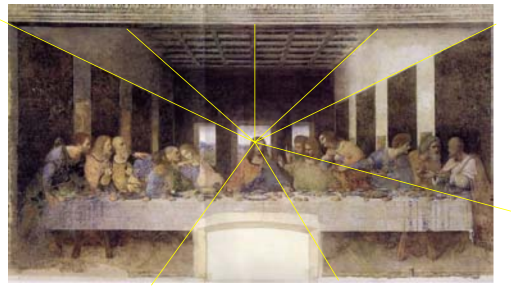


Perspective

- Perspective not only provides a visual structure for the painting but a narrative focus as well.
- Since the eye travels to the vanishing point of a picture, Renaissance artists didn't hesitate to put something important at or near that point.


Piero Della Francesca, *Ideal City*

The Last Supper



The School of Athens by Raphael

The School of Athens by Raphael

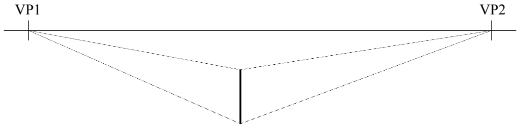
- The *School of Athens* was painted by twenty-seven year-old Raphael Sanzio for Pope Julius II (1503-1513).
- It depicts Plato, Aristotle, Socrates, Pythagoras, Euclid, Alcibiades, Diogenes, Ptolemy, Zoroaster and Raphael.
- Plato is in the center pointing his finger to the heavens while holding the *Timaeus,* his treatise on the origin of the world.

The School of Athens by Raphael

- Next to him, his pupil Aristotle holds a copy of his *Ethics* in one hand and holds out the other in a gesture of moderation, the golden mean.
- Euclid is shown with compass, lower right.
- Pythagoras, Greek philosopher and mathematician, is in the lower-left corner.
- Pythagoras is explaining the musical ratios to a pupil.

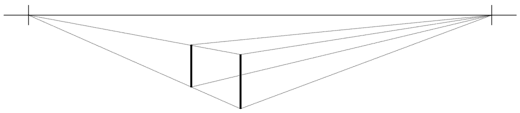
Two Point Perspective

- Draw the horizon line across the top of the paper.
- Mark two vanishing points at either end.
- Draw a vertical line for the “front edge” of the box and then draw convergence lines from the top and bottom of the line to each vanishing point.



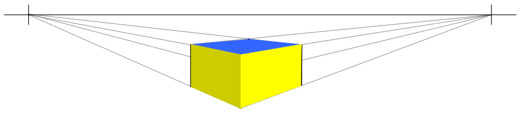
Two Point Perspective

- Next draw a vertical line to the left of your front edge, between the top and bottom construction lines.
- From the top and bottom points of this line, draw construction lines back to the RIGHT vanishing point (VP2).



Two Point Perspective

- Next, draw a similar vertical line to the right of your front edge, and from the top and bottom points of this line, draw construction lines back to the LEFT vanishing point (VP1).
- Where the top construction lines intersect, drop a vertical line to the intersection of the bottom construction lines – this will give you the back edge of the box.
- Erase the construction lines and any obstructed interior lines.



Important Contributors

- Filippo Brunelleschi [1377-1446]
- Leone Battista Alberti [1404-1472]
- Piero della Francesca [1412-1492]
- Albrecht Dürer [1471-1528]
- Leonardo da Vinci [1452-1519]

Filippo Brunelleschi [1377-1446]

- Filippo Brunelleschi was the first great Florentine architect of the Italian Renaissance.
- He began his training in Florence as an apprentice goldsmith in 1392, soon after becoming a master.
- He was active as a sculptor for most of his life and is one of the group of artists, including Alberti, Donatello, and Masaccio, who created the Renaissance style.

Filippo Brunelleschi [1377-1446]

- Brunelleschi’s most important mathematical achievement came around 1415 when he rediscovered the principles of linear perspective using mirrors.
- He understood that there should be a single vanishing point to which all parallel lines in a plane, other than the plane of the canvas, converge.
- He computed the relation between the actual length of an object and its length in the picture depending on its distance behind the plane of the canvas.

Filippo Brunelleschi [1377-1446]

- All of Brunelleschi's works indicate that he possessed inventiveness as both an engineer and as an architect.
- Brunelleschi was the first architect to employ mathematical perspective to redefine Gothic and Romanesque space and to establish new rules of proportioning and symmetry.
- Although Brunelleschi was considered the main initiator of stylistic changes in Renaissance architecture, critics no longer consider him the "Father of the Renaissance."

Filippo Brunelleschi [1377-1446]

- His most notable works:
  - The churches of San Lorenzo and San Spirito
  - The Pazzi Chapel
  - Santa Maria degli Angeli
  - The Pitti Palace
  - The Palazzo Quaratesi
  - Loggia at San Pero a Grada
  - The Cathedral of Florence
  - The Foundling Hospital

Leone Battista Alberti [1404-1472]

- His architectural ideas were the product of his own studies and research.
- Two main architectural writings:
  - *De Pictura* (1435) in which he emphatically declares the importance of painting as a base for architecture and the laws of perspective.
  - *De Re Aedificatoria* (1450) his theoretical masterpiece – It told architects how buildings should be built, not how they were built.

Leone Battista Alberti [1404-1472]

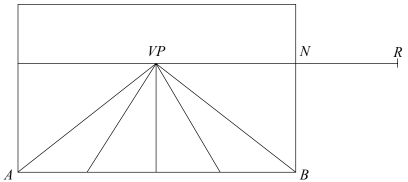
- Alberti studied the representation of 3-dimensional objects.
  - *Nothing pleases me so much as mathematical investigations and demonstrations, especially when I can turn them to some useful practice drawing from mathematics the principles of painting perspective and some amazing propositions on the moving of weights.*
- Alberti also worked on maps and he collaborated with Toscanelli who supplied Columbus with maps for his first voyage.
- He also wrote the first book on cryptography which contains the first example of a frequency table.

Alberti's Construction

- In *De Pictura*, Alberti explains how to construct a tiled floor in perspective.
- First, the vanishing point *VP* is chosen as the point in the picture directly opposite the viewer's eye.
- The ground plane *AB* in the picture is divided equally, and each division point is joined to *VP* by a line.
- These are the convergence lines or orthogonals.

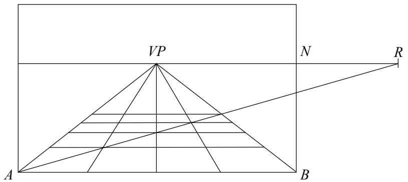
Alberti's Construction

- Next, the "right diagonal vanishing point" *R* is determined by setting *NR* as the "viewing distance."
- The "viewing distance" is how far the painter was from the picture or how far a viewer should stand from the picture.



Alberti's Construction

- Drawing a convergence line from A to R, gives the intersection points where you should draw horizontals parallel to AB.



Piero della Francesca [1412-1492]

- Recognized as one of the most important painters of the Renaissance.
- In his own time he was also known as a highly competent mathematician.

Piero della Francesca [1412-1492]

- Piero showed his mathematical ability at an early age and went on to wrote several mathematical treatises.
- Of these, three have survived:
  - *Abacus treatise (Trattato d'abaco)*
  - *Short book on the five regular solids (Libellus de quinque corporibus regularibus)*
  - *On perspective for painting (De prospectiva pingendi)*.

Piero della Francesca [1412-1492]

- The *Abacus treatise* deals with arithmetic, starting with the use of fractions, and works through series of standard problems, then it turns to algebra, and works through similarly standard problems.
- Finally, geometry where he comes up with some entirely original 3-dimensional problems involving two of the "Archimedean polyhedra" – the truncated tetrahedron and the cuboctahedron.
  - A *cuboctahedron* is a solid which can be obtained by cutting the corners off a cube.
  - It has 8 faces which are equilateral triangles and 6 faces which are squares.

Francesca's *Trattato d'Abaco*

- The Rule of the Three Things states you should multiply the thing which the person wants to know by that which is dissimilar, then divide the result by the other.
- The result is of the nature of that which is dissimilar, and always the divisor is similar to the thing which the person wants to know.
- Example: 7 loaves of bread are worth 9 lire, what will 5 loaves be?

Francesca's *Trattato d'Abaco*

- Multiply the quantity you want to know by the value of 7 loaves of bread, that is, $5 \times 9 = 45$, then divide by 7, and the result is 6 lire, remainder 3 lire.
  - 1 *lira* = 20 *soldi* and 1 *soldo* = 12 *denarii*
- The remainder of 3 lire, gives 60 soldi, divide by 7 yields 8 soldi with a remainder of 4 soldi.
- In denarii, that's 48, divide by 7 gives $6\ {}^{6}/_{7}$ denarii.
- Thus, 5 loaves of bread are worth 6 lire, 8 soldi, and $6\ {}^{6}/_{7}$ denarii.

Francesca’s *Trattato d’Abaco*

- Example: If $3\,{}^1/_3$ loaves of bread cost 15 lire, 2 soldi, 3 denarii. What will 10 loaves cost?
- Multiply 10 by 15 lire, 2 soldi, 3 denarii, getting 151 lire, 2 soldi, 6 denarii.
- This quantity is to be divided by $3\,{}^1/_3$ loaves of bread.
  - Make them whole numbers by multiply by 3
- So we have 453 lire, 7 soldi, 6 denarii divided by 10 loaves of bread.

Francesca’s *Trattato d’Abaco*

- Divide first the lire, which are 453, by 10 you get 45 lire remainder 3 lire.
  - 3 lire = 60 soldi, and 7 makes 67 soldi, divided by 10 gives 6 soldi remainder 7 soldi.
  - 7 soldi = 84 denarii, and the 6 which there are already makes 90, divide by 10 yields 9 denarii.
- Putting it all together you will have 45 lire, 6 soldi, 9 denarii.

Francesca’s *Trattato d’Abaco*

- “*Four companions enter into a partnership; the first enters in the month of January and invests 100 lire, the second enters in April and invests 200 lire, the third enters in July and invests 300 lire, and the fourth enters in October and invests 400 lire; and they stay together until the next January. They have earned 1000 lire, I ask how much each one takes for himself?*”

Francesca’s *Trattato d’Abaco*

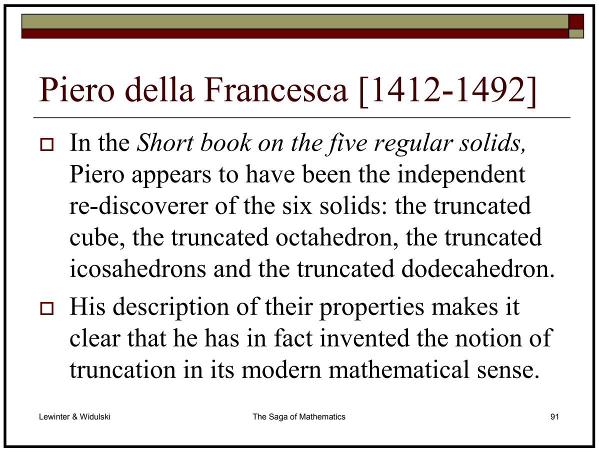
- Suppose first each one earns 2 denarii per lira per month for the time they have been together.
  - The first, who invested 100 lire, has been in the company for one year, at 2 denarii per lira per month, 100 lire earn 10 lire.
  - The second, who has been in the company 9 months and invested 200 lire, at 2 denarii per lira per month, gets 15 lire.
  - The third, who has been in the company 6 months, 300 lire at 2 denarii per month per lira gets 15.
  - The fourth, who has been 3 months, at 2 denarii per month, 400 gets 10 lire.

Francesca’s *Trattato d’Abaco*

- The first gets 10 lire, the second gets 15 lire, the third gets 15 lire, the fourth 10 lire; all together this makes 50, which is the divisor.
- They have earned 1000, to see what each one takes:
  - Multiply 10 by 1000, get 10000, divide by 50 you get 200; so the first one takes 200.
  - For the second, multiply 15 by 1000, get 15000, divide by 50 you get 300; so the second one takes 300.

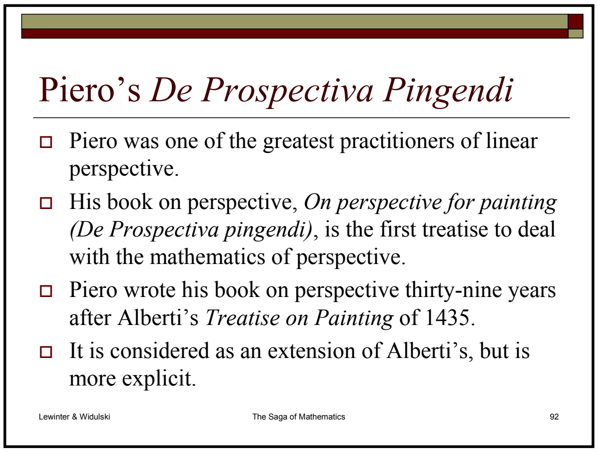
Francesca’s *Trattato d’Abaco*

  - For the third, multiply 15 by 1000, get 15000, divide by 50, you get 300; so the third one takes 300.
  - Multiply 10 by 1000, get 10000, divide by 50 you get 200: so the fourth one takes 200.
- The first takes 200, the third 300, the second 300, the fourth 200.



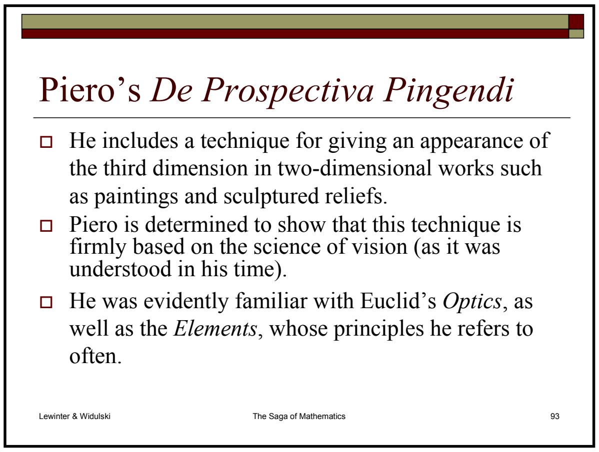
Piero della Francesca [1412-1492]

- In the *Short book on the five regular solids,* Piero appears to have been the independent re-discoverer of the six solids: the truncated cube, the truncated octahedron, the truncated icosahedrons and the truncated dodecahedron.
- His description of their properties makes it clear that he has in fact invented the notion of truncation in its modern mathematical sense.



Piero's *De Prospectiva Pingendi*

- Piero was one of the greatest practitioners of linear perspective.
- His book on perspective, *On perspective for painting (De Prospectiva pingendi)*, is the first treatise to deal with the mathematics of perspective.
- Piero wrote his book on perspective thirty-nine years after Alberti's *Treatise on Painting* of 1435.
- It is considered as an extension of Alberti's, but is more explicit.

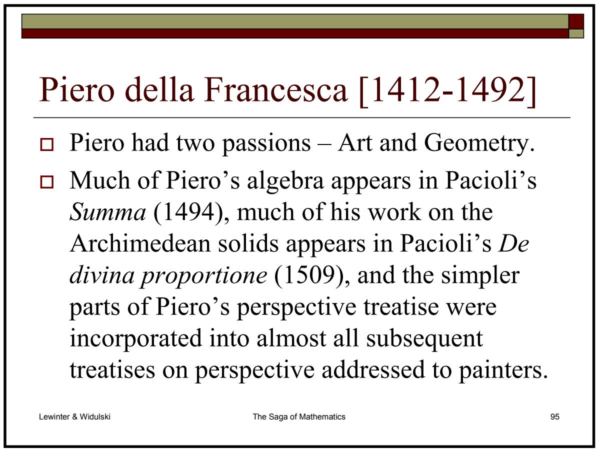


Piero's *De Prospectiva Pingendi*

- He includes a technique for giving an appearance of the third dimension in two-dimensional works such as paintings and sculptured reliefs.
- Piero is determined to show that this technique is firmly based on the science of vision (as it was understood in his time).
- He was evidently familiar with Euclid's *Optics*, as well as the *Elements*, whose principles he refers to often.

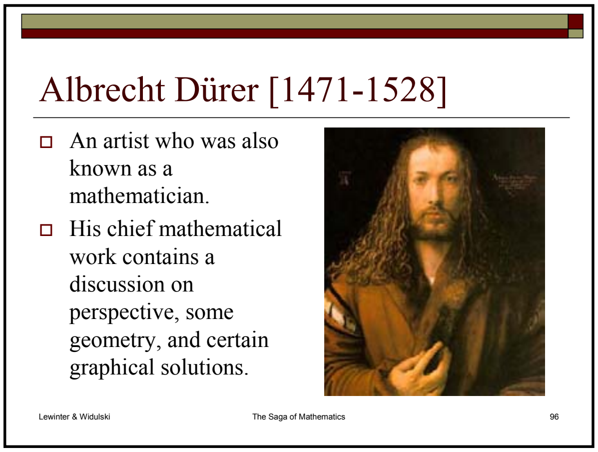
Piero della Francesca [1412-1492]

The Flagellation



Piero della Francesca [1412-1492]

- Piero had two passions – Art and Geometry.
- Much of Piero's algebra appears in Pacioli's *Summa* (1494), much of his work on the Archimedean solids appears in Pacioli's *De divina proportione* (1509), and the simpler parts of Piero's perspective treatise were incorporated into almost all subsequent treatises on perspective addressed to painters.



Albrecht Dürer [1471-1528]

- An artist who was also known as a mathematician.
- His chief mathematical work contains a discussion on perspective, some geometry, and certain graphical solutions.

Albrecht Dürer [1471-1528]

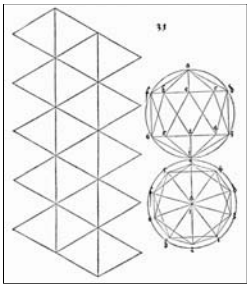
- In 1505, he began an in depth study of measurement, perspective and proportion.
- He believed that mastery of these subjects was fundamental to the improvement and advance of artistic achievement.
- His first publication in 1525, "*Instruction in the Art of Mensuration with Compass and Rule*" contains numerous geometrical figures.

Albrecht Dürer [1471-1528]

- His book contained many interesting curves including the *epicycloid*, the *epitrochoid*, the *hypocycloid*, the *hypotrochoid* and the *limacon*.
- For those who played with a Spirograph as a child you maybe familiar with these curves.
- Check out Spirograph!

Albrecht Dürer [1471-1528]

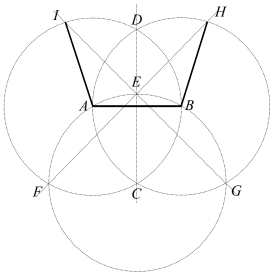
- He showed how to construct regular solids by paper folding.
- This is the 20-sided Platonic solid called the *icosahedron*.
- He also showed how to construct a regular pentagon.



Dürer's Pentagon Construction

- Start with line *AB* and draw two circles one centered at *A,* the other centered at *B,* both with radius *AB*.
- Label their intersections *C* and *D*.
- Draw the line segment *CD* which is the perpendicular bisector of *AB*.
- Next, draw a circle centered at *C* with radius *CA=AB*.
- This circle intersects line *CD* at *E* and the other two circles at *F* and *G*.
- Draw lines through *FE* and *GE* until the intersect the original two circles at *H* and *I*.

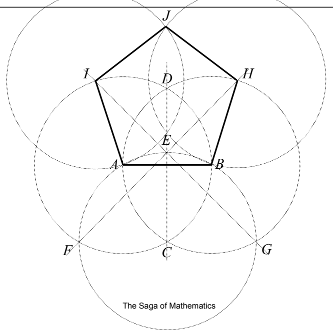
Dürer's Pentagon Construction



Dürer's Pentagon Construction

- This gives us three sides of the pentagon.
- To finish, use the compass to draw a circle at *I* with radius *IA=AB* and one at *H* with radius *HB=AB*.
- Label where they intersect *J*.
- The points *A*, *B*, *I*, *H*, and *J* are the vertices of Dürer's pentagon.

Dürer's Pentagon Construction



Albrecht Dürer [1471-1528]

- In 1514, Albrecht Dürer created an engraving named ***Melancholia*** that included a magic square and some interesting solids.
- Recall, a *magic square* is a square array of numbers 1, 2, 3, ... , n^2 arranged in such a way that the sum of each row, each column and both diagonals is constant.

Albrecht Dürer's Magic Square

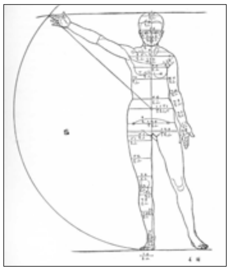
- The number n is called the *order* of the magic square and the constant is called the *magic sum*.
- The *magic sum* is $(n^3 + n)/2$.
- In the bottom row of his 4×4 magic square, he placed the numbers "*15*" and "*14*" side by side to reveal the date of his engraving.

Albrecht Dürer's Magic Square

16	***3***	***2***	***13***
5	***10***	***11***	***8***
9	***6***	***7***	***12***
4	***15***	***14***	***1***

Albrecht Dürer [1471-1528]

- He also wrote *Four Books of Human Proportion*.
- The first two books deal with the proper proportions of the human form; the third changes the proportions according to mathematical rules, giving examples of extremely fat and thin figures, while the last book depicts the human figure in motion.



Leonardo da Vinci [1452-1519]

- Leonardo da Vinci's fame as an artist has overshadowed his claim to consideration as a mathematician.
- His mathematical writings are concerned with mechanics, hydraulics, and optics.

Leonardo da Vinci [1452-1519]

- Between 1482 and 1499, Leonardo was in the service of the Duke of Milan as a painter and engineer.
- He was also considered as a hydraulic and mechanical engineer.
- During his time in Milan, Leonardo became interested in geometry.

Leonardo da Vinci [1452-1519]

- He read Leon Battista Alberti's books on architecture and Piero della Francesca's *On Perspective in Painting.*
- He worked with Pacioli and illustrated Pacioli's *Divina proportione.*
- Allegedly, he neglected his painting because he became so engrossed in geometry.

Leonardo da Vinci [1452-1519]

- Leonardo studied Euclid's *Elements* and Pacioli's *Summa.*
- He also did his own geometry research, sometimes giving mechanical solutions.
- He gave several methods of squaring the circle using mechanical methods.
- He wrote a book on the elementary theory of mechanics.

Leonardo da Vinci [1452-1519]

- In *Codex Atlanticus* written in 1490, Leonardo realized the construction of a telescope and speaks of

  ... making glasses to see the Moon enlarged.
- In *Codex Arundul* written around 1513, he states that

  ... in order to observe the nature of the planets, open the roof and bring the image of a single planet onto the base of a concave mirror. The image of the planet reflected by the base will show the surface of the planet much magnified.

Leonardo da Vinci [1452-1519]

- Leonardo's ideas about the Universe included:
  - He understood the fact that the Moon shone with reflected light from the Sun and he correctly explained the "old Moon in the new Moon's arms" as the Moon's surface illuminated by light reflected from the Earth.
  - He thought of the Moon as being similar to the Earth with seas and areas of solid ground.

False Perspective

- The painting *False Perspective* by William Hogarth foreshadows the work of M. C. Escher.
- Each building has a different vanishing point.
- The smaller objects are closer to the front.

Mathematics and Art

- Mathematics and Art - Perspective
- Mathematics in Art and Architecture
- Art of the Middle Ages
- Geometry in Art and Architecture
- Mathematics and Art Project
- 2003 Mathematics Awareness Month
- Art and Linear Perspective

Mathematics and Art

- Mathematics and Art at www.ams.org
- Drawing: Art Studio Chalkboard
- The World of Escher
- Art by Math gallery
- Symmetry
- Anamorphic Art
- Tessellation Tutorial

Mathematics of the Renaissance

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Mathematics of the Renaissance

- By the middle of the fifteenth century, the mathematical works of the Greeks and Arabs were accessible to European students.
- Dissemination of information became easier with the invention of printing.
- Syncopated algebra and trigonometry.
- The development of symbolic algebra.

Johann Müller [1436-1476]

- Used the name Johann Regiomontanus.
- Took advantage of the recovery of the original texts of the Greek mathematical works.
- He was also well read in the works of the Arab mathematicians.

Johann Müller [1436-1476]

- “You, who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles. Knowing these ideas will open the door to all of astronomy and to certain geometric problems.” – Johann Regiomontanus, from *De Tringulis Omnimodis*.

Johann Müller [1436-1476]

- Made important contributions to trigonometry and astronomy.
- His book *De triangulis omnimodis* (1464) is a systematic exposition of trigonometry, plane and spherical.
  - It is divided into five books.
  - The first four are on plane trigonometry, in particular, determining triangles from three given conditions.

Johann Müller [1436-1476]

- Regiomontanus was the first publisher of mathematical and astronomical books for commercial use.
- In 1472, he made observations of a comet which were accurate enough to allow it to be identified with Halley's comet 210 years later.
- In 1474, he printed his *Ephemerides* containing tables listing the position of the sun, moon, and planets.
- Christopher Columbus had a copy of it on his fourth voyage to the New World.

Nicholas de Cusa [1401-1464]

- Ordained in 1440, he quickly became cardinal and later bishop.
- "A reformer before the reformation."
- He wrote on calendar reform and the squaring of the circle.
- He was interested in geometry and logic and he contributed to the study of infinity.

Nicholas de Cusa [1401-1464]

- His interest in astronomy led him to certain theories which are true and others which may still prove to be true.
- For example:
  - He claimed that the Earth moved round the Sun.
  - He also claimed that the stars were other suns and that space was infinite.
  - He also believed that the stars had other worlds orbiting them which were inhabited.

Luca Pacioli [1454-1514]

- He was a Franciscan Friar.
- He was a renowned mathematician, captivating lecturer, teacher, prolific author, religious mystic, and acknowledged scholar in numerous fields.

Luca Pacioli [1454-1514]

- Piero della Francesca had a studio in the same town in which Pacioli lived.
- Pacioli may have received at least a part of his education there evidenced by the extensive knowledge that Pacioli had of his work.
- He moved to Venice to work, tutor and learn.
- During his time in Venice, Pacioli wrote his first work, a book on arithmetic.

Luca Pacioli [1454-1514]

- He left Venice and traveled to Rome where he spent several months living in the house of Leone Battista Alberti.
- Pacioli travelled, spending time at various universities teaching arithmetic.
- He wrote two more books on arithmetic but none of the three were published.
- Pacioli eventually returned to his home town of Sansepolcro.

Luca Pacioli [1454-1514]

- During this time, Pacioli worked on one of his most famous books the *Summa de arithmetica, geometria, proportioni et proportionalita*.
- In 1494, Pacioli travelled to Venice to publish the *Summa*.
- It was the most influential mathematical book since Fibonacci's *Liber Abaci* and it is notable historically for its wide circulation.

Pacioli's *Summa*

- The earliest printed book on arithmetic and algebra mainly based on Fibonacci's work.
- It consisted of two parts:
  - Arithmetic and algebra
  - Geometry
- The first part gives rules for the four basic operations and a method for extracting square roots.

Pacioli's *Summa*

- Deals fully with questions regarding mercantile arithmetic, in particular, he discusses bills of exchange and the theory of double entry book-keeping.
- This new system was state-of-the-art, and revolutionized economy and business.
- Thus, ensuring Pacioli place as "The Father of Accounting."

Pacioli's *Summa*

- In the section on algebra, he discusses simple and quadratic equations and problems on numbers that lead to such equations.
- He believes that the solution of cubic equations is as impossible as the quadrature of the circle.
- Many of the problems are solved by the "method of false assumption."

Pacioli's *Summa* Example 1

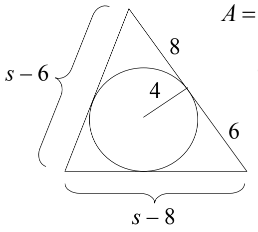
- Find the original capital of a merchant who spent a quarter of it in Pisa and a fifth of it in Venice, who received on these transactions 180 ducats, and who has in hand 224 ducats.
  - Guessing 100 ducats, he spent ¼(100) = 25 and 1/5(100) = 20 or 45 in total, leaving 100–45= 55.
  - Actually, he had 224 – 180 = 44 ducats left.
  - The ratio of his original capital is to 100 ducats as 45 is to 55. Thus, x is to 100 as 44 is to 55.
  - Solving the proportion gives $x = 80$.

Pacioli’s *Summa* Example 2

- Nothing striking in the results in the geometrical part of the work.
- Like Regiomontanus, he applied algebra to aid in investigation of geometrical figures.
  - The radius of an inscribed circle of a triangle is 4 inches and the segments into which the side is divided by the point of contact are 6 inches and 8 inches, respectively. Determine the other sides.

Pacioli’s *Summa*

Using Heron’s Formula



$$A = rs = \sqrt{s(s-a)(s-b)(s-c)}$$

$$4s = \sqrt{s \times (s-14) \times 6 \times 8}$$

$$\Rightarrow s = 21$$

Pacioli’s *Summa*

- The most interesting aspect of the *Summa* is that it studied games of chance.
- Although the solution he gave is incorrect, Pacioli studied the “problem of points.”
- The problem of points is one of the earliest problems that can be classified as a question in probability theory.
- It is concerned with the fair division of stakes between two players when the game is interrupted before the end.

The Problem of Points

- *A team plays ball so that a total 60 points required to win the game and the stakes are 22 ducats. By some accident, they cannot finish the game and one side has 50 points, and the other 30. What share of the prize money belongs to each side*?
- Pacioli’s solution is to divide the stakes in the proportion 5:3, the ratio of points already scored. Does this seem fair to you?

Luca Pacioli [1454-1514]

- Around 1496, the duke of Milan invited Pacioli to teach mathematics at his court where Leonardo da Vinci served as a court painter and engineer.
- Pacioli and da Vinci became friends and discussed mathematics and art at great length.
- Pacioli began writing his second famous work, *Divina proportione*, whose illustrations were drawn by Leonardo da Vinci.

Pacioli’s *Divina Proportione*

- Consisted of three parts, the first of the these studied the “Divine Proportion” or “golden ratio” which is the ratio $a : b = b : (a + b)$.
- It contains the theorems of Euclid which relate to this ratio, and it also studies regular and semiregular polygons.
- The golden ratio was also important in architectural design and this topic is covered in the second part.
- The third was a translation into Italian of one of della Francesca’s works.

Luca Pacioli [1454-1514]

- Pacioli worked with **Scipione del Ferro** and it is conjectured the two discussed the solution of cubic equations.
- Certainly Pacioli discussed the topic in the *Summa* and after Pacioli's visit to Bologna, del Ferro solved one of the two cases of this classic problem.
- Despite the lack of originality in Pacioli's work, his contributions to mathematics are important, particularly because of the influence his books had.

Luca Pacioli [1454-1514]

- The importance of Pacioli's work:
  - His computation of approximate values of square roots (using a special case of Newton's method).
  - His incorrect analysis of games of chance (similar to those studied by Pascal which gave rise to the theory of probability).
  - His problems involving number theory.
  - His collection of many magic squares.

Scipione del Ferro [1465-1526]

- Scipione del Ferro is known for solving the general cubic equation
  $$ax^3 + bx^2 + cx + d = 0.$$
- Whether he solved it himself or discovered it in Arab texts which had made their way to Europe is unclear.
- None of del Ferro's notes have survived.

Scipione del Ferro [1465-1526]

- This is due, at least in part, to his reluctance to make his results widely known.
- Back then mathematicians made money by competing in equation solving contests.
- Thus, by not revealing his secret he could pose questions that only he could solve.
- We do know that he kept a notebook in which he recorded his most important discoveries.

Scipione del Ferro [1465-1526]

- Some say del Ferro began work on the solution after a visit by Pacioli to Bologna.
- The problem of solving the general cubic was reduced to solving the two "depressed" equations:
  1. $x^3 + mx = n$
  2. $x^3 = mx + n$

  where m and n are positive numbers.
- Shortly after Pacioli's visit, del Ferro solved one of the two cases.

The Depressed Equation

- Given the general cubic
  $$ay^3 + by^2 + cy + d = 0,$$
  substitute $x = y - {}^{b}/_{3a}$ and you obtain
  $$x^3 + mx + n = 0$$
  where $m = c - b^2/3a$ and $n = d - bc/3a + 2b^3/27a^2$.
- However, without knowledge of negative numbers, del Ferro would not have been able to use his solution of the one case to solve all cubic equations.

Scipione del Ferro [1465-1526]

- Upon del Ferro death, his notebook passed to his student Antonio Fior.
- Fior was a mediocre mathematician and tried to capitalize on del Ferro's discovery by challenging Tartaglia to a contest.
- Niccolo Tartaglia prompted by the rumors of a solution managed to solve both equations.
- This gave him the advantage in the contest.

Niccolo Fontana Tartaglia [1499-1557]

- Father of ballistics.
- Tartaglia – "the stammerer."
- As a boy, he was wounded when the French captured his home town of Brescia, resulting in a speech impediment.

Tartaglia [1499-1557]

- He could only afford to attend school for fifteen days, but managed to steal a copy of the text and taught himself how to read and write.
- Tartaglia acquired such a proficiency in mathematics that he earned a livelihood by lecturing at Verona.
- Eventually, he was appointed chair of mathematics at Venice.

Tartaglia [1499-1557]

- Most famous for his acceptance of the challenge by Antonio Fior.
- According to this challenge each of them deposited a stake and whoever could solve the most problems out of a collection of thirty proposed by the other would win.
- Fior failed to solve any while Tartaglia could solve them all.

Tartaglia [1499-1557]

- Chief works include:
  - *Nova Scientia* (1537) investigates the laws governing falling bodies and determines that the range of a projectile was maximum when the angle is 45º.
  - *Inventioni* (1546) contains his solution of cubic equation.
  - *Trattato di Numeri et Misure* consists of a treatise on arithmetic (1556) and a treatise on numbers (1560).

Tartaglia [1499-1557]

- In the later, he shows how the coefficients of x in the expansion of $(1 + x)^n$ can be obtained using a "triangle."
- The treatise on arithmetic contains a large number of problems concerning mercantile arithmetic.
- Like Pacioli, Tartaglia included problems concerning mathematical puzzles.

Recreational Mathematics

- *“Three ladies have for husbands three men, who are young, handsome, and gallant, but also jealous. The party are traveling, and find on the bank of a river, over which they have to pass, a small boat which can hold no more than two persons. How can they pass, it being agreed that, in order to avoid scandal, no woman shall be left in the society of a man unless her husband is present?”*

Recreational Mathematics

- *“3 missionaries and 3 obediant but hungry cannibals have to cross a river using a 2-man rowing boat. If on either bank cannibals outnumber missionaries the missionaries will be eaten. How can everyone cross safely?”*

Recreational Mathematics

- *“30 passengers are in a sinking ship. The lifeboat holds 15. They all stand in a circle. Every 9th passenger goes overboard. Where are the 15 lucky positions in the circle?”*

1, 2, 3, 4, 10, 11, 13, 14, 15, 17, 20, 21, 25, 28, and 29.

Recreational Mathematics

- *“Three men robbed a gentleman of a vase containing 24 ounces of balsam. Whilst running away they met in a wood with a glass-seller of whom in a great hurry they purchased three vessels. On reaching a place of safety they wish to divide the booty, but they find that their vessels contain 5, 11, and 13 ounces, respectively. How can they divide the balsam into equal portions?”*

Recreational Mathematics

- The fewest number of steps is 6.

24	13	11	5
24	0	0	0
13	0	11	0
8	0	11	5
8	5	11	0
8	13	3	0
8	8	3	5
8	8	8	0

Recreational Mathematics

- The AIMS Puzzle Corner
- Mathematical Puzzles
- Mathematical Games and Recreations
- Recreational Mathematics at mathschallenge.net
- Recreational Mathematics at www.numericana.com

Girolamo Cardano [1501-1576]

- Cardano was a man of extreme contradiction – the genius closely allied with madness.
- He was an astrologer yet a serious student of philosophy, a gambler yet a first rate algebraist, a physician yet the father of a murderer, a heretic who published the horoscope of Christ yet a recipient of a pension from the Pope.

Girolamo Cardano [1501-1576]

- Girolamo Cardano was the illegitimate child of a lawyer Fazio Cardano whose expertise in mathematics was such that he was consulted by Leonardo da Vinci on questions of perspective and geometry.
- Instead of following in his father's footsteps, Cardano decided to become a doctor – this probably appealed to his hypochondrical nature.

Girolamo Cardano [1501-1576]

- After graduating, he applied to join the College of Physicians in Milan, but was denied due to his being illegitimate.
- Although Cardano practiced medicine without a license, he supported his family by gambling.
- Cardano's understanding of probability meant he had an advantage over his opponents and, in general, he won more than he lost.

Girolamo Cardano [1501-1576]

- Despite his abilities, he ended up in the poorhouse.
- Fortunately, Cardano had a change of luck and became a lecturer in medicine and mathematics at the University of Pavia.
- He continued to practice medicine.
- Eventually, his application to the College of Physicians was accepted in 1539.

Girolamo Cardano [1501-1576]

- In that same year, Cardano published two mathematical books, the second *The Practice of Arithmetic and Simple Mensuration* was a sign of greater things to come.
- Cardano had a prolific literary career writing on a variety of topics including medicine, physics, philosophy, astronomy and theology.
- In mathematics alone, he wrote 21 books, 8 of which were published.

Cardano's *Ars Magna* (1545)

- His *Ars Magna* was the most complete treatise on algebra at that time.
- Unlike other algebraist, Cardano discussed negative and complex roots of equations.
- It contains the solution to the cubic equation that he obtained from Tartaglia under an oath of secrecy and the solution to the quartic equation discovered by his student Ferrari.

Cardano’s *Ars Magna* (1545)

- Cardano presents the first calculation with complex numbers.
- Solve: $x + y = 10$ and $xy = 40$
- This is equivalent to

$$x^2 - 10x + 40 = 0$$

- He showed the solution to be

$$x = 5 + \sqrt{-15} \text{ and } x = 5 - \sqrt{-15}$$

Cardano’s *Liber de Ludo Aleae*

- Published after his death in 1663, it is the first systematic treatment of probability.
- Cardano defined probability as the number of favorable outcomes divided by the total number of possible outcomes.
- Like Tartaglia, he wrote about the error in Pacioli’s solution to the *Problem of the Points*.