Greeks Bearing Gifts

Chapter 4

Ideas or Forms

- This doctrine asserts a view of reality consisting of two worlds:
  1. The everyday world perceived by our senses, the world of change, appearance, and imperfect knowledge.
  2. The world of Ideas perceived by reason, the world of permanence, reality, and true knowledge.
- Justice is an Idea imperfectly reflected in human efforts to be just.
- Two is an Idea participated in by every pair of material objects.

Universals

- A common concern among Greek philosophers, like Plato and Aristotle, was the meaning of *universals*, or *forms*.
- A **universal** can be defined as an abstract object or term which ranges over particular things.
- The classic problem of universals involves whether abstract objects exist in a realm independent of human thought.
- Realists argue that they do.

Plato and Aristotle

- Plato (ca. 428-348 BC), the first and most extreme realist, argued that universals are forms and exist in their own spiritual realm lying outside space and time.
- Individual objects, such as a dog, then participate in the universal form 'dogness'.
- A universal can only be known by the intellect, and not the senses.
- They are timeless perfect patterns of Being, whose blurred shadowy copies constitute the deceptive phenomena of the world around us.

Plato and Aristotle

- Plato's metaphysics and epistemology centered around the concept of the universal:
  - to have knowledge of a particular object, we need to access the unchanging universals.
- The particulars, for Plato, are only manifestations of the forms.
- Plato's theory is subject to the problem of explaining how universals are represented in their particulars and how a universal can reside in a particular.

Plato and Aristotle

- Aristotle (ca. 384-322) criticizes Plato's theory for introducing an aspect of separateness to the universal which was unneeded.
- He also attacks Plato for holding that a universal was a property as well as a substance.
- Aristotle believed that universals did not exist independently of particulars.
- He thought of universals as only being present in the particular things encountered through experience, thus rejecting any concept of "the forms."

Plato and Aristotle

- Aristotle too believed that universals such as "color" exist independently of human thought, but not in a spiritual form-like realm.
- Instead, universals are to be found in the specific shared attributes of individual objects.
- For example, the abstract object "greenness" is found in the class of all green individual objects, such as trees and grass.
- For example, the essence of dog resides in each dog.

Plato and Geometry

- Geometry establishes necessary connections between the forms of the polygons we see around us.
- We don't speak of *this* rectangular table or *that* circular clock on the wall in geometry.
- We discover truths, rather, of the perfect circle or the perfect rectangle.
- These are eternal truths.
- Plato founded a school of philosophy which he named *The Academy*.

Plato and Geometry

- Legend has it that he wrote over its portals "*Let no one destitute of geometry enter my doors*."
- Plato and his school began to emphasize the importance of solid geometry.
- Plato regarded geometry as "the first essential in the training of philosophers", because of its abstract character.
- In *The Republic*, we learn that Plato believes we need a science of solid objects in order to consider issues of objects in motion, such as astronomy.

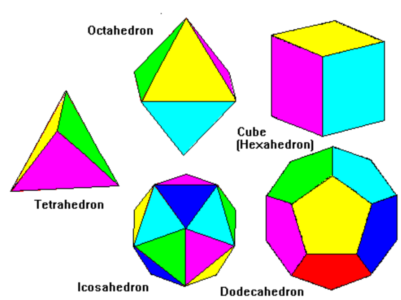
The Platonic Solids

- The Platonic Solids belong to the group of geometric figures called *polyhedra.*
- A **polyhedron** is a solid bounded by plane polygons. The polygons are called **faces**; they intersect in **edges**, the points where three or more edges intersect are called **vertices**.
- A **regular** polyhedron is one whose faces are identical regular polygons.
- Only five **regular solids** are possible:
  - cube (earth), tetrahedron (fire), octahedron (air), icosahedron (water), dodecahedron (universe)

The Platonic Solids

- These are known as **the Platonic Solids**.
- Plato used the five regular polyhedrons in his explanation of the scientific phenomena of the universe.
- Plato was mightily impressed by these five definite shapes that constitute the only perfectly symmetrical arrangements of a set of (non-planar) points in space, and late in life he expounded a complete "**theory of everything**" (in the treatise called *Timaeus*) based explicitly on these five solids.

The Platonic Solids



Euclid of Alexandria (ca. 325-265 BC)

- Not much is known about the personal life of Euclid.
- The little we do know comes from Proclus, the last major Greek philosopher.
- He wrote the most famous and greatest of all textbooks, *The Elements.*

Euclid of Alexandria (ca. 325-265 BC)

- *Elements* is, in large part, not an original work but a compilation of knowledge that became the center of mathematical teaching for 2000 years.

Euclid of Alexandria

- Euclid must have received his mathematical training in Athens (no other place had the manuscripts he studied).
- He must have left by 322 B.C. because the death of Alexander the Great created confusion and turmoil.
- In Alexandria, he found refuge and peace.
- He established a school of mathematics and there wrote his *Elements.*

Euclid of Alexandria

- A few years later, Ptolemy Soter, King of Egypt, founded Museum on a site within his own palace park.
- Museum was supported out of the royal treasury and became the first national university.
- Euclid became Museum's first teacher of mathematics.

Euclid of Alexandria

- Celebrated scholars from all over the world were invited to teach there.
- During the first six centuries of its existence, every noted man of science was either a pupil or teacher, or both, in Museum.
- The Library contained 600,000 papyrus rolls.

Euclid of Alexandria

- Word of mouth about Euclid's lectures in geometry reach King Ptolemy and curious to see the class, Ptolemy spent some time observing it.
- Interested, he asked Euclid to give him instruction.
- The King followed the first few propositions with patience.

Euclid of Alexandria

- Then he interrupted. "The work is most interesting," he said. "Nor do I find it too difficult. But there is the time element to consider. The King has many duties to perform, leaving but little time for a lengthy course in geometry. Is there no shorter road to its mastery?"
- Euclid replied, *"There is no royal road to geometry."*

Euclid of Alexandria

- Another story told by Stobaeus is the following:
  - *... someone who had begun to learn geometry with Euclid, when he had learnt the first theorem, asked Euclid "What shall I get by learning these things?" Euclid called his slave and said "Give him threepence since he must make gain out of what he learns."*
    - T L Heath, *A history of Greek mathematics* **1** (Oxford, 1931).

The Elements

- Although many results in the Elements were not first proved by Euclid the organization of the material and its exposition are certainly due to him.
- The first printed edition appeared in 1482.
- No "best-seller" in history approached Euclid in sales, it was a "must" in all schools and colleges until the beginning of the 20th century.

The Elements

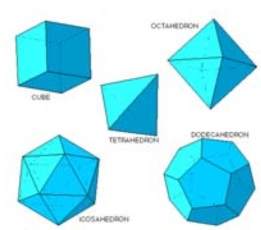
- The Elements consists of thirteen books, with more than 450 propositions.
- Books I to VI deal with plane geometry.
  - Books I and II set out basic properties of triangles, parallels, parallelograms, rectangles and squares.
  - Book III and IV give properties and problems of the circle.
  - Book V lays out the work of Eudoxus on proportion applied to commensurable and incommensurable magnitudes.
  - Book VI looks at applications of the results of Book V to plane geometry.

The Elements

- Books VII to IX deal with number theory.
  - In particular, Book VII is a self-contained introduction to number theory and contains the ***Euclidean algorithm*** for the GCD of two numbers.
  - Book VIII looks at numbers in geometrical progression.
- Book X deals with the theory of irrational numbers.

The Elements

- Books XI to XIII deal with 3-dimensional geometry.
  - The main results of book XII are that circles are to one another as the squares of their diameters and that spheres are to each other as the cubes of their diameters.
  - Book XIII discusses the properties of the five regular polyhedra and gives proof that there are precisely five.



The Elements

- The Elements begins with 23 “definitions.”
- Some of these are not definitions, like “*A point is that which has no part*.”
- Others, like Definition 23, have withstood the test of time.
  - *Parallel straight lines are straight lines which are in the same plane, and if extended indefinitely in both directions, do not meet in either direction.*

The Elements

- Book I contains 10 axioms, divided into two groups of five each.
- Euclid called the first group *postulates* and the second *common notions.*
- Aristotle explained the distinction between these two terms.
  - Common notions are self-evident truths which apply to all sciences.
  - A postulate is an assumption which differs from a axiom in two respects (1) it applies to the subject at hand and (2) it is not self-evident.

Common Notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the parts.

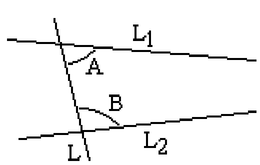
Postulates

1. We can draw a straight line from any point to any point.
2. We can produce a finite straight line continuously in a straight line.
3. We can describe a circle with any center and radius.
4. All right angles are equal to one another.

– and –

The 5th Postulate

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



The 5th Postulate

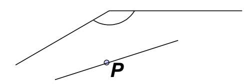
- It is clearly different from the others!
- It is the most convincing evidence of his mathematical genius.
- Proof or No proof!
- Euclid had no proof. In fact, no proof is possible, but he couldn’t go further without this statement.
- Many have tried to prove it from the other nine but failed.

Alternatives to the 5th

- **Poseidonius** (c. 135-51 BC): Two parallel lines are equidistant.
- **Proclus** (c. 500 AD): If a line intersects one of two parallel lines, then it also intersects the other.
- **Saccheri** (c. 1700): The sum of the interior angles of a triangle is two right angles.

Alternatives to the 5th

- **Legendre** (1752-1833): A line through a point in the interior of an angle other than a straight angle intersects at least one of the arms of the angle.



- **Farkas Bólyai** (1775-1856): There is a circle through every set of three non-collinear points.

Playfair's Axiom

- *Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.*
- The Axiom is not Playfair's own invention. He proposed it about 200 years ago, but Proclus stated it some 1300 years earlier.
- It is often substituted for the fifth postulate because it is easier to remember.

Non-Euclidean Geometry

- In the beginning of the last century, some mathematicians began to think along more radical lines.
- Suppose the 5th Postulate is not true!
- *"Through a given point in a plane, <u>two</u> lines, parallel to a given straight line, can be drawn."*
- This would change propositions in Euclid.

Non-Euclidean Geometry

- For example: The sum of the angles in a triangle is *<u>less</u>* than two right angles!
- This of course led to Non-Euclidean geometry whose discovery was led by Gauss (1777-1855), Lobachevskii (1792-1856) and Jonas Bólyai (1802-1850).
- Later by Beltrami (1835-1900), Hilbert (1862-1943) and Klein (1849-1945).
- Fits Einstein's *Theory of Relativity*.

The Theory of Numbers

- In Books 7 to 9 of Elements, two theorems stand out and show the importance of Euclid's work and genius in number theory.
  - The Proof of **The Infinitude of Primes**.

  and
  - **The Euclidean Algorithm** for the GCD of two numbers.

Primes

- A **prime** number is divisible only by 1 and itself.
  - Like 17, whose divisor are only 1 and 17.
- Question: Are there infinitely many prime numbers?
- Euclid gives us the answer.
- Yes, there are infinitely many primes.
- Before we look at Euclid's proof of the **infinitude of primes**, we need some preliminary observations:

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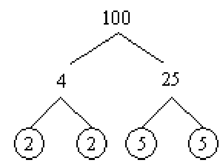
Prime Factorization

- **Every integer can be factored down to prime factors.**
- Example: Write 100 as the product of prime factors.
  - Since $100 = 4 \times 25$ which are not prime factors. Keep factoring. $100 = 2 \times 2 \times 5 \times 5$, and we get our prime factorization!
- This is normally written using a ***factor tree***.



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More Observations

- A number is divisible by another number ***if and only if*** the second number is a factor of the first.
  - 36 is divisible by 12 because $36 = 3 \times 12$.
- If a number, say n, is divisible by another, say m ($m > 1$), then if we add one to the first number, it is no longer divisible by the second, i.e., $n + 1$ is not divisible by m.

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More Observations

- A (rather lengthy) product of many integers is divisible by each of them.
  - $2\times3\times5\times7$ is divisible by each of 2, 3, 5 and 7. (It is divisible by more numbers as well such as 6 and 10.)
- Finally, if a number > 1 is composite (not prime) it must have a prime factor.

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Example of the Proof

- We start with number 3, and prove there is a prime greater than 3. Numbers 6, 9, 12, 15, 18, etc. are all multiples of 3.
- We now add 1 to each multiple of 3 in the sequence above: 7, 10, 13, 16, 19, etc+1!
- The number 3 is not a factor of 7, 10, 13, 16, 19, etc., for 3 is a factor 6, but not of 1 in the sum (6+1), …

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Example of the Proof

- In general, 3 is a factor of all its multiples; but 3 is not a factor of the sum formed by adding 1 to a multiple.
- Such numbers, whose only common factor is 1, are called *relatively prime.*
  - Any multiple of 5, say 55, if increased by 1, gives us a sum 56 of which 5 cannot be a factor (56 and 5 are relatively prime).

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Example of the Proof

- Similarly, 30 is a multiple of 2, 3, and 5.
- Hence, 31 is prime to 2, 3, and 5.
- We will now arrange, in the form of a table, the material needed for the proof that there is a prime greater than 17.

Successive Primes	Prod	Sum
2 · 3	6	7
2 · 3 · 5	30	31
2 · 3 · 5 · 7	210	211
2·3·5·7·11	2,310	2311
2·3·5·7·11·13	30,030	30031
2·3·5·7·11·13·17	510,510	510511

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Example of the Proof

- The multiple of all primes on line 6 is 510,510.
- These primes factors are only factors of 510,510.
- We now add 1 to this multiple, giving 510,511.
- This sum is either prime or composite (not prime).

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Example of the Proof

- If 510,511 is prime, we found one larger than 17.
- If 510,511 is composite, and not only divisible by 1 and itself, but by other number as well, it has a prime factor different from the all those primes which we used to build up 510,510, that is, all the primes from 2 to 17.
- Therefore, if it is composite, it has a prime factor greater than 17!
- The algebraic proof is very similar…

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Euclid's Theorem

- ***Theorem:*** *There are infinitely many primes.*
- ***Proof***: It suffices to show that if p is any given prime, then there exists a prime > p. Let 2, 3, 5, …, p be the complete list of primes up to p. If we form the number $N = (2\times3\times5\times\cdots\times p)+1$, then it is clear that $N>p$ and also that N either is itself a prime or it is divisible by some prime $q<N$, and in the latter case the preceding remarks imply that $q>p$. Thus, in each case there exists a prime > p, and this is what we set out to prove.

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Euclidian Algorithm

- Euclid developed an *algorithm* (procedure) for determining the **greatest common divisor (GCD)** of two integers.
  - Example: the GCD of 15 and 20 is 5 because 5 is the greatest (or largest) number which goes into both 15 and 20.
  - We write this as the GCD(15, 20) = 5.
- He noticed that if a number n goes into x and into y, then it also goes into $x - y$.

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Euclidian Algorithm

- He reasoned that we can subtract the smaller number from the larger as many times as possible.
- Then compute the GCD of the remainder and the smaller number.
- Yielding a much easier problem.
- Let's look at some examples!

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Euclidian Algorithm Example

- <u>Example</u>: Find the GCD of 30 and 650.
  - Observe that we may subtract 30 (many times) from 650 till we get a remainder of 20.
  - A quick way to do this is to divide 30 into 650, discarding the quotient, 21, and keeping the remainder of 20.) So, 650 ÷ 30 = 21 *R* 20.
  - Then the GCD of the original pair, 30 and 650, must go into 20.

Example (continued)

- Since it also goes into 30, we can simply compute the GCD of 20 and 30, which by guesswork is 10.
- Congratulations! This is the GCD of the original pair.
- If guesswork were inadequate here, we would repeat the procedure by dividing 20 into 30 and retaining the remainder of 10.
- The GCD of 20 and 10 is 10, which we already know is the answer.

Another Example

- <u>Example</u>: Find the GCD of 48 and 360.
  - Divide the larger number by the smaller number and discard the quotient.
  - 360 ÷ 48 = 7 *R* 24, so we discard the 7 and keep the 24.
  - We do this because the GCD of 48 and 360 is the same as the GCD of 24 and 48.
  - So, now we find the GCD(48, 24).

Example (continued)

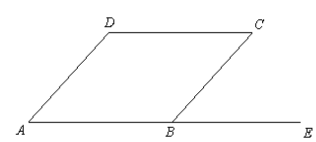
- To find the GCD(24, 48), we divide again, (that is if you don't see the answer yet), and we obtain 48 ÷ 24 = 2 *R* 0.
- When we get a remainder of zero, the algorithm stops.
- When this happens, the answer is the last divisor, in this case 24.
- Thus, the GCD(360, 48) = GCD(48, 24) = 24!

Euclidian Geometry

- Let's look at a typical theorem in The Elements.
- *<u>Theorem</u>: The adjacent angles of a parallelogram are supplementary, that is, they add up to 180º.*
- The proof involves extending side *AB* to an arbitrary point *E* in the Figure.

Euclidian Geometry

- *AE* is a transversal across parallel lines *AD* and *BC*, thus $\angle DAE = \angle CBE$.
- Because *ABE* is a straight line, $\angle ABC + \angle CBE = 180^\circ$,
- Substituting $\angle DAE$ for $\angle CBE$, gives $\angle ABC + \angle DAE = 180^\circ$.
- *Q.E.D.*



Forgotten Euclid

- Elements completely overshadowed Euclid's other writings.
  - ***Data*** (with 94 propositions) - looks at what properties of figures can be deduced when other properties are given.
  - ***On Divisions*** - looks at constructions to divide a figure into two parts with areas of given ratio.
  - ***Optics*** - is the first Greek work on perspective.
  - ***Phaenomena*** - is an elementary introduction to mathematical astronomy and gives results on the times stars in certain positions will rise and set.

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Lost Euclid

- Euclid's following books have all been lost:
  - ***Surface Loci*** (two books)
  - ***Porisms*** (a three book work with, according to Pappus, 171 theorems and 38 lemmas),
  - ***Conics*** (four books)
  - ***Book of Fallacies*** and
  - ***Elements of Music***

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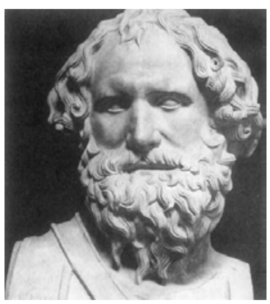
Archimedes (ca. 287-212 BC)

- Archimedes was a native of Syracuse, Sicily.
- The achievements of Archimedes are quite outstanding.
- He is considered one of the greatest mathematicians of all time.



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Archimedes (ca. 287-212 BC)

- Studied at the Museum at Alexandria.
- Was a contemporary of Eratosthenes (ca. 276-194 BC) who made a surprisingly accurate measurement of the circumference of the Earth but is best known for his sieve for finding primes.
- Famous throughout the Greek world during his lifetime and has been a legendary figure ever since, for his mathematical writings, mechanical inventions, and the brilliant way he conducted the defense of his native city during the Second Punic War (218-201 BC).

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Archimedes (ca. 287-212 BC)

- In one story, he was asked by Hieron II to determine whether a crown was pure gold or was alloyed with silver.
- Archimedes observed the overflow of water in his bath, he suddenly realized that since gold is more dense (i.e., has more weight per volume) than silver, a given weight of gold represents a smaller volume than an equal weight of silver.
- Thus, a given weight of gold would displace less water than an equal weight of silver.
- Delighted at his discovery, he ran home without his clothes, shouting "*Eureka*!"

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Archimedes' Law of Buoyancy

- States that a any object, wholly or partly immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.
- An object will float when the buoyancy force equals the weight of the object.
- The principle applies to both floating and submerged bodies and to all fluids, i.e., liquids and gases.

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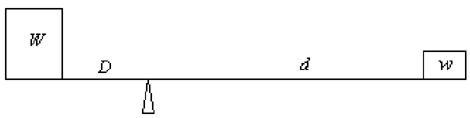
Archimedes’ Law of the Lever

- States that $W \times D = w \times d$.
- For example, a 140 lb. boy 2 feet from the fulcrum (center of gravity) balances his 70 lb. sister 4 feet from the fulcrum.
- *“Give me a place to stand and I will move the earth.”*



Archimedes

- Archimedes proved, among many other geometrical results, that the volume of a sphere is two-thirds the volume of a circumscribed cylinder.
- The surface area of the sphere is four times the area of a great circle of the sphere.
- He invented the whole study of hydrostatics, static mechanics, pycnometry (the measurement of the volume or density of an object).
- He is considered the “father of integral calculus.”

Archimedes’ Works

- His works which have survived are:
  - *On the Equilibrium of Planes* (two books)
  - *On the Quadrature of the Parabola*
  - *On the Sphere and Cylinder* (two books)
  - *On Spirals*
  - *On Conoids and Spheroids*
  - *On Floating Bodies* (two books)
  - *On the Measurement of the Circle*, and
  - *Arenarius* [*The Sandreckoner*]
- In 1906, J L Heiberg, discovered a 10th century manuscript of Archimedes’ work *The method.*

Archimedes’ Works

- *On the Quadrature of the Parabola* is a treatise of 24 propositions that contains two proofs of his theorem that the area of a segment of a parabola, i.e., the region cut from a parabola by any traverse line, is 4/3 the area of the triangle with the same base and height.
- He also sums infinite series in such a way as to demonstrate his awareness of the concept of a limit.

Archimedes’ Works

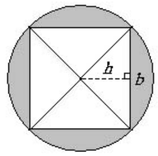
- *On Spirals* is a treatise of 28 propositions about the curve known today as the “*spiral of Archimedes*.”
- *On the Sphere and Cylinder* contains rigorous proofs of his great discoveries of their volume and surface area.
- *On Conoids and Spheroids* which he means solids of revolution generated by revolving parabolas, hyperbolas, and ellipses about their axes, he discoveries many elementary integral calculus formulas.

On the Measurement of the Circle

- One of his greatest achievements was his conquest of the circle.
- Shows that the area of a circle is equal to that of a triangle with base equal to its circumference and height equal to its radius, i.e., $A = \frac{1}{2} \times C \times r$.
- Since $C = 2 \times \pi \times r$, we get $A = \pi r^2$.
- Gives an approximation of π, $3 \, ^{10}/_{71} < \pi < 3 \, ^{1}/_{7}$ – which he reached after circumscribing and inscribing a circle with two 96-sided polygons.

On the Measurement of the Circle

- The 'inscribed' square has sides of length *b*.
- The distance from the center of the circle to the midpoint of each side is length *h*.
- Thinking of the square as composed of four triangles, it has an area of $4\times(\frac{1}{2}\times b\times h) = \frac{1}{2}\times(4\times b)\times h$.
- Note that $4\times b$ is the perimeter of the square.



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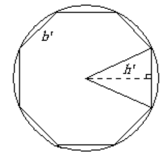
On the Measurement of the Circle

- Next, Archimedes inscribed an octagon with side *b'*.
- The distance from the circle's center to the midpoint of each side is *h'*.
- Thinking of the square as composed of eight triangles, the area is $8\times(\frac{1}{2}\times b'\times h') = \frac{1}{2}\times(8\times b')\times h'$.
- Note that $8\times b'$ is the perimeter of the octagon.



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On the Measurement of the Circle

- In both cases the area of the inscribed polygon is
$$A = \frac{1}{2}\times P\times h.$$
- Archimedes wondered what would happen if this doubling process could be continued forever – or *ad infinitum*.
- The perimeters of these figures get closer to the '*limiting*' value C which is the circumference of the circle, and the distances from the center to the midpoints of the sides would approach *r*, the length of the radius of the circle.

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On the Measurement of the Circle

- Thus, he discovered the beautiful formula $A = \frac{1}{2}\times C\times r$, in which A represents the exact area of the circle.
- Now all circles have the same shape, that is, they are similar.
- The ratio of the circumference of any circle to its diameter is constant.
- This ratio is called π.

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On the Measurement of the Circle

- Calling the diameter d, we have $C/d = \pi$, or $C = \pi\times d$.
- Since the diameter is twice as long as the radius, we write $C = 2\pi r$.
- Substituting $2\pi r$ for C in the formula
$$A = \frac{1}{2}\times C\times r$$
- and simplifying, we obtain the world famous
$$A = \pi r^2.$$

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Archimedes' Inventions

- Archimedes' claw
- The catapult
- The compound pulley
- Burning mirrors
- Archimedean screw - an early type of pump which is still used in traditional agriculture in some areas of the world.

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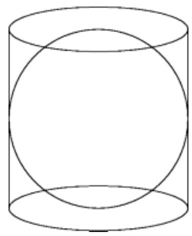
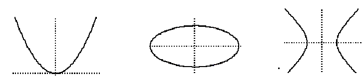
Archimedes' Tombstone

- Archimedes requested that his tombstone be decorated with a sphere contained in the smallest possible cylinder and inscribed with the ratio of the cylinder's volume to that of the sphere, which is 3/2.
- Archimedes considered the discovery of this ratio the greatest of all his accomplishments.



Apollonius of Perga (ca. 262-190 BC)

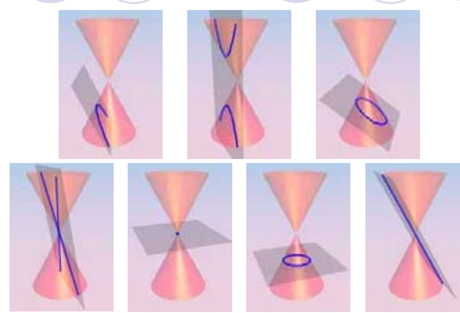
- Known as "The Great Geometer."
- His famous book *On Conics* had a great influence on the development of mathematics and introduced the terms which we use today such as *parabola*, *ellipse* and *hyperbola*.



Apollonius' *On Conics*

- Books I-IV contain elementary properties of conic sections. Much of which was known but he organized and arranged it anew.
- Books V-VII seem to contain discoveries which he himself had made.
  - Book V is on normals and tangents to a conic section.
  - Book VI is on the equality and similarity on conics.
  - Book VII on diameters and rectilinear figures described on these diameters
- Book VIII is lost!

Conic Sections



Apollonius of Perga

- Apollonius took geometric constructions beyond that of Euclid's *Elements.*
- In the *Elements* Book III, Euclid shows how to draw a circle through three given points.
- He also shows how to draw a circle tangent to three given lines.
- In *Tangencies* Apollonius shows how to construct the circle which is tangent to three given circles.
- More generally he shows how to construct the circle which is tangent to any three objects, where the objects are points or lines or circles.

The Problem of Apollonius

- Given three things, each of which may be either a point, a straight line, or a circle, draw a circle that is tangent to each, that is, draw a circle which passes through each of the given points (when points are given) and touches the straight lines or circles.
- There are a total of ten cases. The two easiest involve three points or three lines and the hardest involves three circles.
- Euclid had presented the constructions for three points and for three lines. Apollonius present a construction for all of the other cases.

Apollonius of Perga

- Apollonius obtained an approximation for π better than the $223/71 < \pi < 22/7$ known to Archimedes.
- Without Apollonius' *Conics*, Kepler would never have been able to discover his laws of planetary motion, one of which states that the orbit of each planet around the sun is an ellipse with the sun at one focus. (1609)
- Without Kepler's Laws, Newton would probably have been unable to formulate his theory of universal gravitation and his laws of motion. (1687)

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Claudius Ptolemy (ca. 85-165 AD)

- Made a map of the ancient world in which he employed a coordinate system very similar to the latitude and longitude of today.
- His most important work, the *Almagest,* is a treatise in thirteen books.

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Claudius Ptolemy (ca. 85-165 AD)

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Claudius Ptolemy (ca. 85-165 AD)

- He made astronomical observations from Alexandria in Egypt during the years 127-141 AD.
- The *Almagest* gives in detail the mathematical theory of the motions of the Sun, Moon, and planets.
- Ptolemy used geometric models to predict the positions of the sun, moon, and planets, using combinations of circular motion known as *epicycles*.

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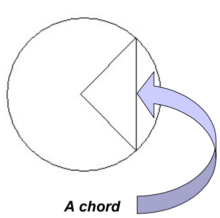
Claudius Ptolemy (ca. 85-165 AD)

- Having set up this model, Ptolemy then describe the mathematics which he needs in the rest of the work.
- One of his greatest achievements was his geometric calculation of *semi-chords*.
- A **chord** of a circle is a line segment with both ends on the circumference of a circle.



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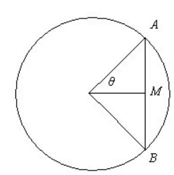
Claudius Ptolemy (ca. 85-165 AD)

- Adding the line segment from the center to the midpoint of the chord, we get two congruent right triangles.
- Now in a right triangle, the sine of an acute angle, abbreviated **sin** θ, is defined as the ratio of the length of the opposite leg to the length of the hypotenuse.



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Claudius Ptolemy (ca. 85-165 AD)

- Under the assumption that the radius equals 1, we get that **sin** θ = the length of segment *AM*, which is exactly half the entire chord *AB*, hence the name *semi-chord*.
- In particular, he introduced trigonometrical methods based on the chord function *Crd* (which is related to the sine function by sin θ = (*Crd* 2θ)/120).
- Using √3 = chord 60°, he obtained √3 = 1.73205.

Ptolemy's Value of π

- He obtained, using chords of a circle and an inscribed 360-gon an approximation for π = 3 + 17/120 = 3.1417.
- Ptolemy actually wrote the value in sexagesimal notation as 3;8,30.
- So this says that π equals 3 + 8/60 + 30/3,600, and this value is within .003 percent of the correct value of π.

Ptolemy's Epicycles

- Ptolemy used geometrical models to explain planetary theory.
- Ptolemy used Apollonius' system of *epicycles* and *eccentric* circles to explain the apparent motion of the planets across the sky.
  - *Eccentric theory* is the theory that planets move round in circles whose centers do not coincide with the Earth.
  - An *epicycle* is a circle whose centre is carried around the circumference of another circle. Epicycles were introduced originally to explain the orbits of planetary bodies among the fixed stars.

Claudius Ptolemy (ca. 85-165 AD)

- Ptolemy required approximately 80 equations to describe, quite accurately, the locations of all the heavenly bodies of what we now call the solar system.
- Ptolemy's geocentric model went unchallenged till the middle of the 16th century and was defended by the church for another few hundred years.
- A heliocentric (sun at the center) system first appeared in print in 1548 in a work written by Copernicus.