Those Incredible Greeks!

Chapter 3

Greece

- In 700 BC, Greece consisted of a collection of independent city-states covering a large area including modern day Greece, Turkey, and a multitude of Mediterranean islands.
- The Greeks were great travelers.
- Greek merchant ships sailed the seas, bringing them into contact with the civilizations of Egypt, Phoenicia, and Babylon.

Greece

- Also brought cultural influences like Egyptian geometry and Babylonian algebra and commercial arithmetic.
- Coinage in precious metals was invented around 700 BC and gave rise to a money economy based not only on agriculture but also on movable goods.
- This brought Magna Greece (“greater Greece”) prosperity.

Greece

- This prosperous Greek society accumulated enough wealth to support a leisure class.
- Intellectuals and artists with enough time on their hands to study mathematics for its own sake, and generally, seeking knowledge for its own sake.
- They realized that non-practical activity is important in the advancement of knowledge.

Greece

- As noted by David M. Burton in his book The History of Mathematics,
  - “*The miracle of Greece was not single but twofold—first the unrivaled rapidity and variety and quality of its achievement; then its success in permeating and imposing its values on alien civilizations.*”

The Greeks

- Made mathematics into one discipline.
- More profound, more rational, and more abstract (more remote from the uses of everyday life).
- In Egypt and Babylon, mathematics was a tool for practical applications or as special knowledge of a privileged class of scribes.

The Greeks

- Made mathematics a detached intellectual subject for the connoisseur instead of being monopolized by the powerful priesthood.
- They weren't concerned with triangular fields, but with "triangles" and the characteristics of "triangularity."
- The Greeks had a preference for the abstract.

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The Greeks

- Best seen in the attitude toward the square root of 2.
  - The Babylonians computed it with high accuracy
  - The Greeks proved it was irrational
- Changed the nature of the subject of mathematics by applying reasoning to it ⇒ **Proofs!**
  - Mathematical 'truths' must be proven!
  - Mathematics builds on itself.

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The Greeks

- Plato's inscription over the door of his academy, "*Let no man ignorant of geometry enter here.*"
- The Greeks believed that through inquiry and logic one could understand their place in the universe.
- The rise of Greek mathematics begins in the sixth century BC with Thales and Pythagoras.
- Later reaching its zenith with Euclid, Archimedes, and Apollonius.
- Followed by Ptolemy, Pappus, and Diophantus.

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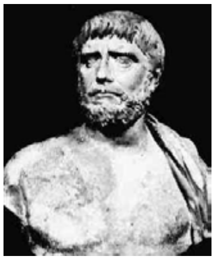
Thales of Miletus

- Born in Miletus, and lived from about 624 BC to about 547 BC.
- Thales was a merchant in his younger days, a statesman in his middle life, and a mathematician, astronomer, and philosopher in his later years.
- Extremely successful in his business ventures.



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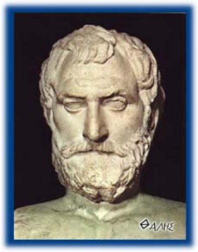
Thales of Miletus

- Thales used his skills to deduce that the next season's olive crop would be a very large one.
- He secured control of all the oil presses in Miletus and Chios in a year when olives promised to be plentiful, subletting them at his own rental when the season came.



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Thales

- Traveled to Egypt, and probably Babylon, on commercial ventures, studying in those places and then bringing back the knowledge he learned about astronomy and geometry to Greece.
- He is hailed as the first to introduce using logical proof based on deductive reasoning rather than experiment and intuition to support an argument.

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Thales

- Proclus states,
  - “*Thales was the first to go into Egypt and bring back this learning [geometry] into Greece. He discovered many propositions himself and he disclosed to his successors the underlying principles of many others, in some cases his methods being more general, in others more empirical.*”

Thales

- Founded the Ionian (Milesian) school of Greek astronomy.
- Considered the father of Greek astronomy, geometry, and arithmetic.
- Thales is designated as the first mathematician.
- The first of the Seven Sages of Greece.

Thales

- His philosophy was that *“Water is the principle, or the element, of things. All things are water.”*
- He believed that the Earth floats on water and all things come to be from water.
- For him the Earth was a flat disc floating on an infinite ocean.

Thales

- “Know thyself” and “nothing overmuch” were some of Thales philosophical ideas.
- Asked what was most difficult, he said, “To know thyself.”
- Asked what was easiest, he answered, “To give advice.”

Thales

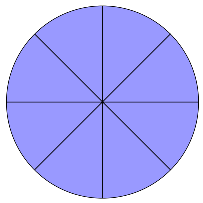
- Thales is credited with proving six propositions of elementary geometry:
  1. *A circle is bisected by its diameter.*
  2. *The base angles of an isosceles triangle are equal.*
  3. *If two straight lines intersect, the opposite angles are equal.*
  4. *Two triangles are congruent if they have one side and two adjacent angles equal.*
  5. *The sides of similar triangles are proportional.*
  6. *An angle inscribed in a semicircle is a right angle. (*)*

Thales

- Thales measured the height of pyramids.
  - Thales discovered how to obtain the height of pyramids and all other similar objects, namely, by measuring the shadow of the object at the time when a body and its shadow are equal in length.
- Thales showed how to find the distances of ships from the shore necessarily involves the use of this theorem (iv).

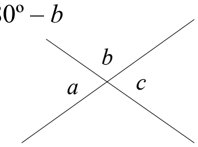
A circle is bisected by its diameter

- Thales supposedly **demonstrated** that a circle is bisected by its diameter.
- But Euclid did not even prove this, rather he only **stated** it.
- It seems likely that Thales also only stated it rather than **proving** it.



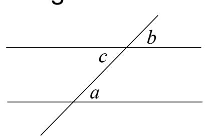
Vertical Angles Are Equal

- *The angles between two intersecting straight lines are equal.*
- $a + b = 180^\circ \Rightarrow a = 180^\circ - b$
- $b + c = 180^\circ \Rightarrow c = 180^\circ - b$
- $\therefore\ a = c.$



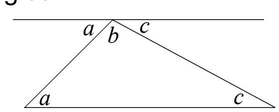
Alternate Interior Angles

- Thales immediately drew forth new truths from these six principles. He observed that a line crossing two given parallel lines makes equal angles with them.



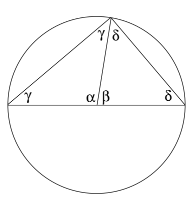
Interior Angles in a Triangle

- The sum of the angles of any triangle is 180º.
- Draw a line through the upper vertex parallel to the base obtaining two pairs of alternate interior angles.
- $\therefore\ a + b + c = 180^\circ.$



Thales

- *An angle in a semicircle is a right angle.*
- $\alpha + \beta = 180^\circ$
- $2\gamma + \alpha = 180^\circ$
- $2\delta + \beta = 180^\circ$
- $2(\delta + \gamma) + (\alpha + \beta) = 360^\circ$
- $\therefore\ \delta + \gamma = 90^\circ.$

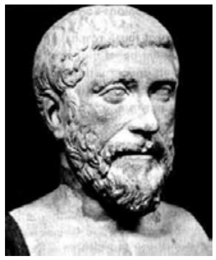


Thales

- One night, Thales was gazing at the sky as he walked and fell into a ditch.
- A pretty servant girl lifted him out and said to him "*How do you expect to understand what is going on up in the sky if you do not even see what is at your feet.*"

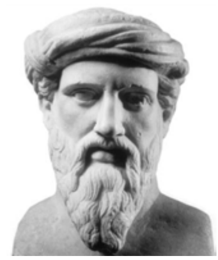
Pythagoras of Samos

- Little is known about the life of Pythagoras.
- He was born about 569 BC on the Aegean island of Samos.
- Died about 475 BC.
- Studied in Egypt and Babylonia.



Pythagoras of Samos

- Pythagoras founded a philosophical and religious school in Croton (now Crotone, on the east of the heel of southern Italy) that had many followers.



Pythagorean Brotherhood

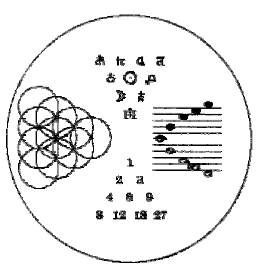
- Pythagoras was the head of the society with an inner circle of followers known as *mathematikoi*.
- The *mathematikoi* lived permanently with the Society, had no personal possessions and were vegetarians.
- They were taught by Pythagoras himself and obeyed strict rules.

Pythagoreans

- Both men and women were permitted to become members of the Society, in fact several later women Pythagoreans became famous philosophers.
- The outer circle of the Society were known as the *akousmatics* (listeners) and they lived in their own houses, only coming to the Society during the day.
- The members were bound not to disclose anything taught or discovered to outsiders.

The Quadrivium

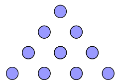
- They studied the "Quadrivium"
  - Arithmetica (Number Theory)
  - Harmonia (Music)
  - Geometria (Geometry)
  - Astrologia (Astronomy)



The Seven Liberal Arts

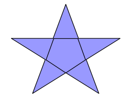
- Later added the "Trivium",
  - Logic
  - Grammar
  - Rhetoric
- Forming the "Seven Liberal Arts"

Pythagoreans



- The symbol they swore their oath on was called the "tetractys".
- It represented the four basic elements of antiquity: fire, air, water, and earth.
- It was represented geometrically by an equilateral triangle made up of ten dots and arithmetically by the sum
  - 1 + 2 + 3 + 4 = 10

Pythagoreans



- The five pointed star, or pentagram, was used as a sign so Pythagoreans could recognize one another.
- Believed the soul could leave the body, I.e., transmigration of the soul.
- "*Knowledge is the greatest purification*"
- Mathematics was an essential part of life and religion.

Pythagoras' Philosophy

- Theorized that everything, physical and spiritual, had been assigned its allotted number and form.
- "*Everything is number.*"
- According to Aristotle, "*The Pythagoreans devoted themselves to mathematics, they were the first to advance this study and having been brought up in it they thought its principles were the principles of all things.*"

Why?

- Music
  - He discovered that notes sounded by a vibrating string depended on the string's length.
  - Harmonious sounds were produced by plucking two equally taut strings whose lengths were in proportion to one another.
  - Proportions related to the tetractys!

Harmonious Musical Intervals

- If one string was twice as long as the other, i.e., their lengths were in ratio 1:2, then an **octave** sounded.
- If their lengths were in ratio 2:3, then a **fifth** sounded.
- If their lengths were in ratio 3:4, then an **fourth** sounded.

Hear Pythagoras' Intervals at http://www.aboutscotland.com/harmony/prop.html

Pythagoras' Astronomy

- An extension of the doctrine of harmonious intervals.
- Each of the 7 known planets (which included the Sun and Moon) was carried around the Earth on its own crystal sphere.
- Each body would produce a certain sound according to its distance from the center.
- Producing a celestial harmony, "*The Music of the Spheres*."

Pythagorean Doctrine

- Mixture of cosmic philosophy and number mysticism.
- A supernumerology that assigned to everything material or spiritual a definite integer.
- They believed that mathematics was the key to the nature of all things and that mathematics was everywhere.

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Numerology

- A mystical belief that was common in many ancient societies.
- Various numbers represented things like love, gender, and hate.
- Even numbers were female while odd numbers were male.
- The number 1 was the omnipotent One and the generator of all numbers.
- The number 2 was the first female number and represented diversity.

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Numerology

- 3 = 1 + 2 was the first male number composed of unity and diversity.
- 4 = 2 + 2 was the number for justice since it is so well balanced.
- 5 = 2 + 3 was the number of marriage.
- Earth, air, water and fire, were composed of hexahedrons, octahedrons, icosahedrons, and pyramids – geometric solids differing in the *number* of faces.

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Classification of Numbers

- He observed that some integers have many factors while others have relatively few.
- For example,
  - The factors of 12 are 1, 2, 3, 4, and 6. (He didn’t consider the number a factor of itself, i.e., he only considered ***proper*** factors.)
  - The proper factors of 10 are 1, 2, and 5. (Another word for factor is *divisor*.)
- Pythagoras decided to compare a number with the sum of its divisors.

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Deficient Numbers

- A number is *deficient* if the sum of its proper divisors is less than the number itself.
- For example,
  - The proper divisors of 15 are 1, 3, and 5.
  - The sum 1 + 3 + 5 = 9 < 15.
  - Therefore, 15 is deficient.

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Abundant Numbers

- A number is *abundant* if the sum of its proper divisors is greater than the number itself.
- For example,
  - The proper divisors of 12 are 1, 2, 3, 4, and 6.
  - The sum 1 + 2 + 3 + 4 + 6 = 16 > 12.
  - Therefore, 12 is abundant.

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Perfect Numbers

- A number is *perfect* if the sum of its proper divisors is equal to the number itself.
- For example,
  - The proper divisors of 6 are 1, 2, and 3.
  - The sum 1 + 2 + 3 = 6.
  - Therefore, 6 is perfect.
  - There are more, 28, 496, and 8128 are all perfect!

Finding Perfect Numbers

- In Euclid's "*Elements*", he presents a method for finding perfect numbers.
- Consider the sums:
  - 1 + 2 = 3
  - 1 + 2 + 4 = 7
  - 1 + 2 + 4 + 8 = 15
  - 1 + 2 + 4 + 8 + 16 = 31
  - 1 + 2 + 4 + 8 + 16 + 32 = 63
  - 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127

Finding Perfect Numbers

- The sums 3, 7, 15, 31, 63 and 127 are each one less than a power of two.
- Looking at the sums 3, 7, 15, 31, 63 and 127, notice that the sums 3, 7, 31 and 127 are *prime* numbers.
- Euclid noticed that when the sum is a prime number, if you multiply the sum by the last power of two in the sum, you get a perfect number!

Perfect Numbers

- 1 + 2 = 3
- 1 + 2 + 4 = 7
- 1 + 2 + 4 + 8 = 15
- 1 + 2 + 4 + 8 + 16 = 31
- 1 + 2 + 4 + 8 + 16 + 32 = 63
- 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127

- 3 x 2 = 6
- 7 x 4 = 28
- 15 is not prime
- 31 x 16 = 496
- 63 is not prime
- 127 x 64 = 8128

Pythagorean Theorem

- Pythagoras' Theorem claims that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse.
- In algebraic terms, $a^2 + b^2 = c^2$ where c is the hypotenuse while a and b are the sides of the triangle.
- A Pythagorean triple is a set of three positive integers (a,b,c) that satisfy the equation $a^2 + b^2 = c^2$.

Pythagorean Triples

- For example, (3, 4, 5) and (5, 12, 13) are Pythagorean triples, so are (6, 8, 10) and (15, 36, 39).
- We make a distinction between them.
- Triples that contain no common factors, like (3, 4, 5) and (5, 12, 13), are called **primitive** Pythagorean triples.

Primitive Pythagorean Triples

- Take two numbers p and q that satisfy:
  1. $p > q$,
  2. p and q have different *parity* (i.e. one is even and the other is odd), and
  3. p and q have no common divisor except 1.

$$a = p^2 - q^2$$
$$b = 2pq$$
$$c = p^2 + q^2$$

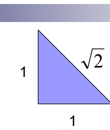
Examples

- ***EXAMPLE****: Find the Pythagorean triple for the generators p = 2 and q = 1.*
  - *Using the equations for a, b and c we get*
  - $a = 2^2 - 1^2 = 3$, $b = 2\times2\times1 = 4$, and $c = 2^2 + 1^2 = 5$.
  - *Wow! We get the beautiful triple 3, 4, 5.*
- ***EXAMPLE****: Find the Pythagorean triple for the generators p = 3 and q = 2.*
  - *Using the equations for a, b and c we get*
  - $a = 3^2 - 2^2 = 5$, $b = 2\times3\times2 = 12$, and $c = 3^2 + 2^2 = 13$.
  - *Amazing! This is the famous 5, 12, 13 triple.*

Rationals

- The Greeks believed that starting with integer lengths like 7 and 38, and then subdividing them into fractions like 7/3 and 38/9, they could express any length.
- We call such quantities the *rational* numbers, because they are ratios of integers.
- For example, $3\frac{1}{2} = \frac{7}{2}$, $0.13 = \frac{13}{100}$, and $2.4 = \frac{24}{10}$

Irrationals



- The Pythagoreans realized that this cannot be done for some numbers, i.e., some numbers are *irrational*.
- They encountered their first irrational in the hypotenuse of a simple right triangle whose legs are both 1.
- The Greeks called these lengths 1 and $\sqrt{2}$ *incommensurable*, meaning that they cannot equal the same length multiplied by (different) whole numbers.

The Irrationality of $\sqrt{2}$

- Basic Facts:
  1. The ratio of two integers can always be reduced to lowest terms.
  2. Squaring a number preserves the *parity* of that number.
  3. The ratio of two odd numbers may or may not be in lowest terms, while the ratio of two even numbers is never in lowest terms.

The Irrationality of $\sqrt{2}$

- Proof by contradiction
- Assume that $\sqrt{2}$ is a rational number and it is a/b, reduced to lowest terms, i.e., a and b have no common divisor.
- Squaring both sides and multiplying both sides by b^2 yields the last equation which implies that a^2 is even.

$$\sqrt{2} = \frac{a}{b}$$
$$2 = \frac{a^2}{b^2}$$
$$2b^2 = a^2$$

The Irrationality of $\sqrt{2}$

- Basic Fact #2 says that a must also be even, implying that a is divisible by 2.
- Then it is 2 times something, i.e., $a = 2m$ for some integer m.
- Then $a^2 = (2m)^2 = 2m2m = 4m^2$.
- Substituting $4m^2$ for a^2 in the last equation $2b^2 = a^2$ on the previous slide, we get $2b^2 = 4m^2$.
- Dividing both sides by 2 yields $b^2 = 2m^2$.

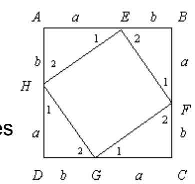
The Irrationality of $\sqrt{2}$

- This implies that b^2 is even and therefore so is b.
- Where are we then? It seems that both a and b are even.
- But didn't we say that the fraction a/b was reduced to its lowest terms.
- This is impossible by Basic Fact #3 and we have obtained a contradiction. Thus, the original assumption – that it was rational – must be false.
- $\sqrt{2}$ is an irrational number!

Irrationals and the Infinite

- The simple geometrical concept of the diagonal of a square defies the integers and negates the Pythagorean philosophy.
- We can construct the diagonal geometrically, but we cannot measure it in any finite number of steps.
- The square root of two can be calculated to any required *finite* number of decimal places (like 1.414), but the decimal never repeats nor terminates.

Proof of Pythagoras' Theorem



$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

Equating these two equations gives

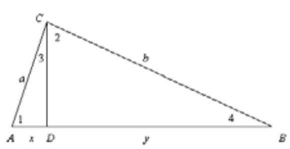
$$a^2 + 2ab + b^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

Subtracting the $2ab$ from both sides gives

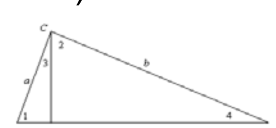
$$a^2 + b^2 = c^2$$

Proof #2

- In right triangle ΔABC, the altitude CD is *perpendicular* to (makes a 90º angle with) hypotenuse AB.
- AD and DB have lengths x and y which add up to c, the length of the hypotenuse, i.e., $c = x + y$.



Proof #2 (continued)



- ∠1 = ∠2
- ∠3 = ∠4
- All three right triangles are similar so certain ratios are equal.
- By comparing triangles ΔACD and ΔABC, we get $a/x = c/a$.
- Comparing triangles ΔBCD and ΔABC, gives $b/y = c/b$.

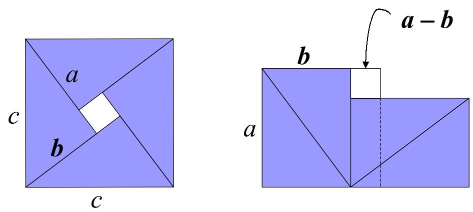
Cross multiplying gives

$a^2 = cx$

$b^2 = cy$

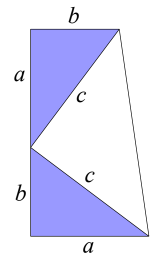
Adding these two equations gives

Proof #3



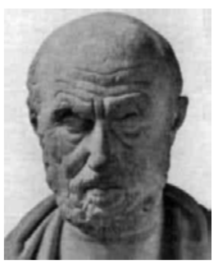
Proof #4 (by President Garfield)

- In 1876, President Garfield discovered his own proof of the Pythagorean Theorem.
- The key is using the formula for the area of a trapezoid - *half sum of the bases times the altitude* – $(a+b)/2 \cdot (a+b)$.
- Looking at it another way, this can be computed as the sum of areas of the three triangles – $ab/2 + ab/2 + c \cdot c/2$.
- As before, simplifications yield $a^2+b^2=c^2$.



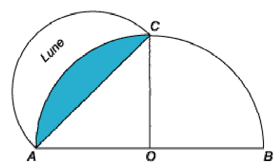
Hippocrates of Chios

- His work entitled *Elements of Geometry* was the first to arrange the propositions of geometry in a scientific fashion.
- In working on the squaring the circle and duplicating the cube, He discovered the area of various *lunes* – regions bounded by arcs of two circles.



Hippocrates of Chios

- The ***lune*** of is obtained by drawing a semicircle with center at *O* and radius *AO*, of length 1.
- The diameter *AB* has length 2.
- Draw radius *OC* such that it is perpendicular to *AB*.
- Δ*AOC* is a right triangle with legs of length 1 and hypotenuse *AC* of length $\sqrt{2}$.



Hippocrates of Chios

- Knew that the areas of two circles were proportional to the squares of their diameters.

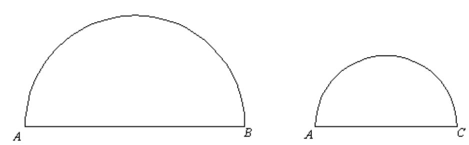
$$\frac{\text{area of semicircle on } AB}{\text{area of semicircle on } AC} = \frac{AB^2}{AC^2}$$

- This ratio must equal 2, since the Pythagorean Theorem gives

$$AB^2 = (2AO)^2 = 4AO^2 = 2(AO^2 + OC^2) = 2AC^2$$

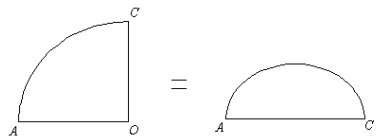
Hippocrates of Chios

- Hence, the semicircle on AB has twice the area of the semicircle on AC.



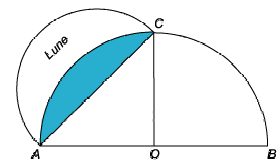
Hippocrates of Chios

- Thus, half the larger has the same area as the smaller.
- We can equate the areas of the semicircle with diameter AC and the quarter-circle (AOC).



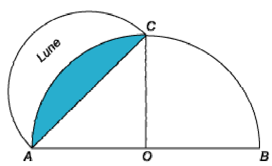
Hippocrates of Chios

- The semicircle and quarter-circle overlap in the shaded segment with corners at *A* and *C*.
- If we remove this overlap from the semicircle and quarter-circle, the leftovers must have the same area.



The Quadrature of the Lune

- But the leftovers are the **lune** with corners at *A* and *C* and the right triangle *AOC*.
- Since the base and height of this right triangle have length one, its area is ½.
- Then this is also the exact area of the lune!

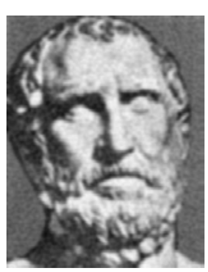


The Eleatic School

- The Eleatic school was founded by the religious thinker and poet Xenophanes.
- The greatest of the Eleatic philosophers was Parmenides.
- His philosophy of monism claimed that the many things which appear to exist are merely a single eternal reality which he called **Being**.
- In other words, the universe is singular, eternal, and unchanging.

Zeno of Elea (ca. 495-435 BC)

- Zeno was a pupil/friend of Parmenides.
- Their principle was that "*all is one*" and that change or non-Being are impossible.
- The appearances of multiplicity, change, and motion are mere illusions.
- Zeno is best known for his paradoxes concerning motion.



Zeno's Paradoxes

- *The Dichotomy*: There is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on ad infinitum.
- *The Achilles*: The slower will never be overtaken by the quicker, for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must always be some distance ahead.

Zeno's Paradoxes

- *The Arrow*: If everything is either at rest or moving when it occupies a space equal to itself, while the object moved is always in the instant, a moving arrow is unmoved.
- *The Stadium*: Consider two rows of bodies, each composed of an equal number of bodies of equal size. They pass each other as they travel with equal velocity in opposite directions. Thus, half a time is equal to the whole time.

Zeno of Elea

- *The Dichotomy*: Motion is impossible!
  - An object moving from point *A* to point *B* must first get to the midpoint; let's call this point *C*.
  - Before the object can reach point *C*, it would have to get to the midpoint between *A* and *C*.
  - Let's call this new point *D*. This argument may be repeated ad infinitum, from which Zeno concluded that motion was impossible.
- It requires traversing infinitely many points in a finite amount of time.

Democritus (ca. 460-370 BC)

- Known as the laughing philosopher.
- Best known for the *atomic theory* of matter, that is, the theory that matter and space are not infinitely divisible.
- Stated that motion was possible by positing the existence of ultimate indivisible particles, called *atoms*, out of which all things are constructed.
- He asserted that one couldn't continue to subdivide something indefinitely.

Democritus (ca. 460-370 BC)

- Discovered theorems in solid geometry:
  - The volume of a cone is one-third the volume of a cylinder having the same base and equal height.
  - The volume of a pyramid is one-third the volume of a prism having the same base and equal height.

Democritus

- Wrote over 75 works on almost every subject, from physics and mathematics to logic, ethics, magnets, fevers, diets, agriculture, law, "the sacred writings in Babylon," "the right use of history," and even the growth of animals, horns, spiders, and their webs, and the eyes of owls.
- Was the Aristotle of the 5th century; and his views have led many to consider him the equal, and perhaps the superior, of Plato.

Democritus

- Plato felt that his writings should be burned, perhaps because of his boastful comments.
  - "*I have wandered over a larger part of the earth than any other man of my time, inquiring about things most remote; I have observed very many climates and lands and have listened to many learned men; but no one has ever yet surpassed me in the construction of lines with demonstration; no, not even the Egyptian rope-stretchers with whom I lived five years in all, in a foreign land.*"

Democritus

- All of his writings except fragments have perished:
  - "*It is hard to be governed by one's inferior.*"
  - "*It is better to examine one's own faults than others.*"
  - "*Many very learned men have no intelligence.*"
  - "*To a wise man the whole earth is his home.*"
  - "*A life without festivity is a long road without an inn.*"

Today

- Both Zeno and Democritus wrestled with a problem that would not be solved for two thousand years.
- The problem of infinitesimal magnitudes.
- Mathematicians today understand that a finite quantity can be represented as a sum of infinitely many progressively smaller quantities.
- An easy example of this is given by the infinite, repeating decimal .999... which equals 1.

Ideas for Papers

- Three Construction Problems of Antiquity.
  - The Problem of Squaring the Circle.
  - The Delian Problem (The Duplication of the Cube).
  - The Problem of Trisecting an Angle.
- The Quadratrix of Hippias of Elis.
- Pierre Wantzel's (1814-1848) proof of the impossibility of these problems.

Ideas for Papers

- Zeno's paradoxes or Democritus' atomic theory.
- The mathematics of Plato and the Platonic number and solids.
- The Greek mathematicians Eudoxus (ca. 408-355 BC), Archytas of Tarentum (ca. 428-350 BC) or Menaechmus (ca. 380-320 BC).
- The Method of Exhaustion.