Mesopotamia Here We Come

Chapter 2

Babylonians

- The Babylonians lived in Mesopotamia, a fertile plain between the Tigris and Euphrates rivers.
- Babylonian society replaced both the Sumerian and Akkadian civilizations.
- The Sumerians built cities, developed a legal system, administration, a postal system and irrigation structure.
- The Akkadians invaded the area around 2300 BC and mixed with the Sumerians.

Babylonians

- The Akkadians invented the abacus, methods for addition, subtraction, multiplication and division.
- The Sumerians revolted against Akkadian rule and, by 2100 BC, had once more attained control.
- They developed an abstract form of writing based on *cuneiform* (i.e. wedge-shaped) symbols.
- Their symbols were written on wet clay tablets which were baked in the hot sun and many thousands of these tablets have survived to this day.

Babylonians

- It was the use of a stylus on a clay medium that led to the use of *cuneiform* symbols since curved lines could not be drawn.
- Around 1800 BC, Hammurabi, the King of the city of Babylon, came into power over the entire empire of Sumer and Akkad, founding the first Babylonian dynasty.
- While this empire was not always the center of culture associated with this time in history, the name Babylonian is used for the region of Mesopotamia from 2000 BC to 600 BC.

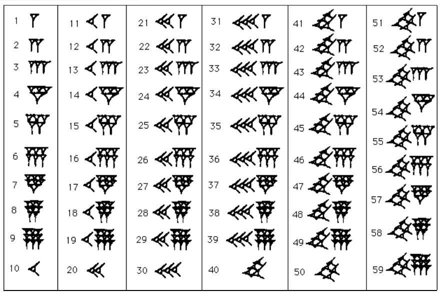
Babylonian Cuneiform

- Because the Latin word for “wedge” is *cuneus*, the Babylonian writing on clay tablets using a wedge-shaped stylus is called **cuneiform**.
- Originally, deciphered by a German schoolteacher Georg Friedrich Grotefend (1775-1853) as a drunken wager with friends.
- Later, re-deciphered by H.C. Rawlinson (1810-1895) in 1847.
- Over 300 tablets have been found containing mathematics.

Babylonian Cuneiform

- Babylonians used a positional system with base 60 or the sexagesimal system.
- A positional system is based on the notion of place value in which the value of a symbol depends on the position it occupies in the numerical representation.
- For numbers in the base group (1 to 59), they used a simple grouping system

Babylonian Cuneiform Numbers 1 to 59

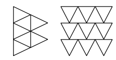


Picture from http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Babylonian_numerals.html.

Lewinter & Widulski The Saga of Mathematics 7

Babylonian Numerals

- We will use for 10 and for 1, so the number 59 is



- For numbers larger than 59, a "digit" is moved to the left whose place value increases by a factor of 60.
- So 60 would also be .

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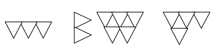
Babylonian Numerals

- Consider the following number



- We will use the notation $(3, 25, 4)_{60}$.
- This is equivalent to

$$3 \times 60^2 + 25 \times 60 + 4 = 12{,}304$$

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Babylonian Numerals

- Drawbacks:
  - The lack of a sexagesimal point
  - Ambiguous use of symbols
  - The absence of zero, until about 300 BC when a separate symbol was used to act as a placeholder.
- These lead to difficulties in determining the value of a number unless the context gives an indication of what it should be.

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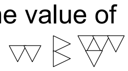
Babylonian Numerals

- To see this imagine that we want to determine the value of



- This could be any of the following:

$$2 \times 60 + 24 = 144$$

$$2 \times 60^2 + 24 \times 60 = 8640$$

$$2 + \frac{24}{60} = 2\frac{2}{5}$$

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Babylonian Numerals

- The Babylonians never achieved an absolute positional system.
- We will use 0 as a placeholder, commas to separate the "digits" and a semicolon to indicate the fractional part.
- For example, $(25, 0, 3; 30)_{60}$ will represent

$$25 \times 60^2 + 0 \times 60 + 3 + \frac{30}{60} = 90{,}003\frac{1}{2}$$

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More Examples

- (25, 0; 3, 30)$_{60}$ represents

$$25\times 60+0+\frac{3}{60}+\frac{30}{60^2}=1500\frac{7}{120}$$

- (10, 20; 30, 45)$_{60}$ represents

$$10\times 60+20+\frac{30}{60}+\frac{45}{60^2}=620\frac{41}{80}$$

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More Examples

- (5; 5, 50, 45)$_{60}$ represents

$$5+\frac{5}{60}+\frac{50}{60^2}+\frac{45}{60^3}=5\frac{1403}{14400}$$

- Note: Neither the comma (,) nor the semicolon (;) had any counterpart in the original Babylonian cuneiform.

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Babylonian Arithmetic

- Babylonian tablets contain evidence of their highly developed mathematics
- Some tablets contain squares of the numbers from 1 to 59, cubes up to 32, square roots, cube roots, sums of squares and cubes, and reciprocals.
- See Table 1 in *The Saga of Mathematics* (page 29)

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Babylonian Arithmetic

- For the Babylonians, addition and subtraction are very much as it is for us today except that carrying and borrowing center around 60 not 10.
- Let's add (10, 30; 50)$_{60}$ + (30; 40, 25)$_{60}$

$$\begin{array}{r} 10,30;50,\ \ 0 \\ +\quad 30;40,25 \\ \hline 11,\ 1;30,25 \end{array}$$

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Babylonian Arithmetic

- Remember to line the numbers up at the sexagesimal point, that is, the semicolon (;) and add zero when necessary.
- Note that since 40 + 50 = 90 which is greater than 60, we write 90 in sexagesimal as (1, 30)$_{60}$.
- So we put down 30 and carry the 1.
- Similarly for the 30 + 30 + 1 (that we carried).

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Babylonian Multiplication

- Some tablets list the multiples of a single number, *p*.
- Because the Mesopotamians used a sexagesimal (base 60) number system, you would expect that a multiplication table would list all the multiples from 1*p*, 2*p*, ..., up to 59*p*.
- But what they did was to give all the multiples from 1*p* up to 20*p*, and then go up in multiples of 10, thus finishing the table with 30*p*, 40*p* and 50*p*.

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Babylonian Multiplication

- They would then use the distributive law $a \times (b + c) = a \times b + a \times c$
- If they wanted to know, say, 47*p*, they added 40*p* and 7*p*.
- Sometimes the tables finished by giving the square of the number *p* as well.
- Since they had tablets containing squares, they could also find products another way.

Babylonian Multiplication

- Using tablets containing squares, the Babylonians could use the formula

$$ab = [(a+b)^2 - a^2 - b^2] \div 2$$

- Or, an even better one is

$$ab = [(a+b)^2 - (a-b)^2] \div 4$$

Babylonian Multiplication

10	1,40	19	6,1
11	2,1	20	6,40
12	2,24	21	7,21
13	2,49	22	8,4
14	3,16	23	8,49
15	3,45	24	9,36
16	4,16	25	10,25
17	4,49	26	11,16
18	5,24	27	12,9

- Using the table at the right, find 11×12.
- Following the formula, we have 11×12 = $(23^2 - 1^2) \div 4$ = $(8, 48)_{60} \div 4$ = $(2, 12)_{60}$.

Babylonian Multiplication

- Multiplication can also be done like it is in our number system.
- Remember that carrying centers around 60 not 10.
- For example, $10\frac{5}{6} \times 30\frac{1}{3}$

$$\begin{array}{r} 10;50 \\ \times \quad 30;20 \\ \hline 3,36,40 \\ +\ 5,25,\ 0 \\ \hline 5,28;36,40 \end{array}$$

Babylonian Division

- Correctly seen as multiplication by the reciprocal of the divisor.
- For example,
  $2 \div 3 = 2 \times (1/3) = 2 \times (0;20)_{60} = (0;40)_{60}$
- For this purpose they kept a table of reciprocals (see Table 1, page 29).
- Babylonians approximated reciprocals which led to repeating sexagesimals.

Babylonian Division

- $44 \div 12 = 44 \times (1/12) = 44 \times (0;5)_{60} = (3;40)_{60}$.
  - Note: 5 × 44 = 220 and 220 in base-60 is 3,40.
- $12 \div 8 = 12 \times (1/8) = 12 \times (0;7,30)_{60} = (1;30,0)_{60}$.
- $25 \div 9 = 25 \times (1/9) = 25 \times (0;6,40)_{60} = (2;46,40)_{60}$.

Babylonian Division

- When fractions generated repeating sexagesimals, they would use an approximation.
- Since $1/7 = (0;8,34,17,8,34,17,...)_{60}$.
- They would have terminated it to approximate the solution and state that it was so, "*since 7 does not divide*".
- They would use $1/7 \cong (0;8,34,17,8)_{60}$.

Lewinter & Widulski The Saga of Mathematics 25

Babylonian Algebra

- Babylonian could solve linear equations, system of equations, quadratic equations, and some cubics as well.
- The Babylonians had some sort of theoretical approach to mathematics, unlike the Egyptians.
- Many problems were intellectual exercises which demonstrate interesting numerical relations.

Lewinter & Widulski The Saga of Mathematics 26

Linear Equations

- *I found a stone but did not weigh it; after I added to it $^1/_7$ of its weight and then $^1/_{11}$ of this new weight, I weighed the total 1 mina. What was the original weight of the stone?*
  - [Note: 1 mina = 60 sheqels and 1 sheqel = 180 se.]
- Answer: $^2/_3$ mina, 8 sheqels, 22 ½ se.
- Or 48.125 sheqels!

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Linear Equations

- Call the original weight *x*, and solve

$$\left(x + \frac{1}{7}x\right) + \frac{1}{11}\left(x + \frac{1}{7}x\right) = 60$$

- This can be reduced to

$$\frac{96}{77}x = 60$$

- To solve, multiply both sides by the reciprocal, $x = (5/8)\times 77$

Lewinter & Widulski The Saga of Mathematics 28

Simultaneous Equations

- *There are two silver rings; $^1/_7$ of the first and $^1/_{11}$ of the second ring are broken off, so that what is broken off weighs 1 sheqel. The first diminished by its $^1/_7$ weighs as much as the second diminished by its $^1/_{11}$. What did the silver rings weigh?*
- Answer: 4.375 sheqels and 4.125 sheqels.

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Simultaneous Equations

- Consider the system

$$\frac{x}{7} + \frac{y}{11} = 1, \quad \frac{6x}{7} = \frac{10y}{11}$$

- This can be solved by substitution.
- Multiply both sides of the first equation by 6, gives

$$\frac{6x}{7} + \frac{6y}{11} = 6$$

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Simultaneous Equations

- Substituting yields

$$\frac{10y}{11} + \frac{6y}{11} = 6 \quad \Rightarrow \quad \frac{16y}{11} = 6$$

- Multiply both sides by the reciprocal

$$y = \frac{11}{16} \times 6 = \frac{33}{8} = 4.125$$

- Using this value in the second equation gives *x*.

Quadratic Equations

- *I have added the area and two-thirds of the side of my square and it is 0;35. What is the side of the square?*
- They solved their quadratic equations by the method of "completing the square."
- The equation is

$$x^2 + \frac{2}{3}x = \frac{35}{60}$$

Completing the Square

- *You take 1 the coefficient [of x]. Two-thirds of 1 is 0;40. Half of this, 0;20, you multiply by 0;20 and it [the result] 0;6,40 you add 0;35 and [the result] 0;41,40 has 0;50 as its square root. The 0;20, which you multiplied by itself, you subtract from 0;50, and 0;30 is [the side of] the square.*
- Amazing! But what is it really saying?

Completing the Square

- To solve: $x^2 + ax = b$
- Take ½ of the coefficient of x.
- Square it.
- Add the right-hand side to it.
- Square root this number.
- Finally subtract ½ of the coefficient of x.

$$x = \sqrt{\left(\frac{a}{2}\right)^2 + b} - \frac{a}{2}$$

Quadratic Formula

- Today, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- This formula is derived by completing the square on the general quadratic equation:

$$ax^2 + bx + c = 0$$

Square Roots

- The Babylonians had an accurate and simple method for finding square roots.
- The method is also known as Heron's method, after the Greek mathematician who lived in the first century AD.
- Also known as Newton's method.
- Indian mathematicians also used a similar method as early as 800 BC.

Square Roots

- The Babylonians are credited with having first invented this square root method in 1900 BC.
- The Babylonian method for finding square roots involves dividing and averaging, over and over, obtaining a more accurate solution with each repetition of the process.

Square Root Algorithm

1. Make a guess.
2. Divide your original number by your guess.
3. Find the average of these numbers.
4. Use this average as your next guess and repeat the algorithm three times.

An Example

- Let's try to find the square root of 37.
- We know a good guess is 6.
- So using the method, we divide the original number by the guess.

  37/6 = 6.1666666666666…
- Find the average of the two numbers.

  (6 + 6.1666…)/2 = 6.0833333333…

An Example

- Use this average as the next guess and repeat the algorithm three times.

  37/6.0833… = 6.0821917808219178…
- (6.08333…+ 6.0821917…)/2 =

  6.0827625570776255707...
- Repeating a third time yields

  6.0827625302982197479479476906083

Percentage Error

- The answer obtained using the calculator on the computer is:

  6.0827625302982196889996842452021
- If we calculate the percentage error, that is,
  - Take the difference in the answers (the error).
  - Divide that by the actual answer, and then
  - Multiply the result by 100.

Percentage Error

- We can see that this method gives an error of approximately

  $5.894826344 \times 10^{-17}$
- The percentage error is

  $9.69103481383 \times 10^{-16}$
- Their formula yields a result that is accurate to 15 decimal places.
- Not bad for 2000 B.C.!!!!

Square Root of 2 (YBC 7289)

- The side of the square is labeled 30 or $(0;30)_{60} = ½$.
- The diagonal is labeled $(1;24,51,10)_{60}$ = 1.4142129 on top and $(0;42,25,35)_{60}$ = 0.7070647 below.

Square Root of 2

- Comparing these numbers with √2 = 1.414213562… and 1/√2 = 0.707106... we can see that the tablet represents a sophisticated approximation to √2 and its reciprocal.
- We can arrive at their approximation if we use their method with an initial guess of 3/2.

Babylonian Geometry

- The Babylonians were aware of the link between algebra and geometry.
- They used terms like length and area in their solutions of problems.
- They had no objection to combining lengths and areas, thus mixing dimensions.

Babylonian Geometry

- They were familiar with:
  - The formulas for the area of a rectangle, right triangles, isosceles triangle, trapezoid, and parallelograms.
  - The Pythagorean Theorem.
  - The proportionality of the sides of similar triangles.
  - The fact that in an isosceles triangle, the line joining the vertex to the midpoint of the base is perpendicular to the base.

Babylonian Geometry

- Babylonian tablets have been found in which they used the value 3 for π.
- They estimated the circumference of a circle as 3 times the diameter, $C = 3 \times d$.
- The area of the circle as $A = C^2/12$.
- The Babylonians also had an estimate of π equivalent to $(3; 7, 30)_{60}$ which is equal to 3.125.

Babylonian Geometry

- Many problems dealt with lengths, widths, and area.
- Given the semi-perimeter $x + y = a$ and the area $xy = b$ of the rectangle. Find the length and width.
- Given the the area and the difference between the length and width. Find the length and width.

Babylonian Geometry

- *The length exceeds the width by 10. The area is 600. What are the length and width?*
- We would solve this by introducing symbols.
- Let x = the length and y = the width, then the problem is to solve:

$$x - y = 10 \text{ and } xy = 600$$

Babylonian Solution

1. Take half the difference of the length and width (the *half-difference*): 5
2. Square the half-difference: 25
3. Add the area: 625
4. Take the *square root*: 25
5. The answers are:

length = square root + half-difference = 30
width = square root – half-difference = 20

Plimpton 322

- Catalog #322 in the *G. A. Plimpton* collection at Columbia University.
- Dated around 1900 to 1600 BC.
- Unfortunately, a piece on the left hand edge has broken off.

Plimpton 322

- The most mathematically significant of the Mesopotamian tablets.
- Proves that the Babylonians knew about the Pythagorean Theorem more than a thousand years before Pythagoras was born.
- Remember, the Pythagorean Theorem says $a^2 + b^2 = c^2$

Plimpton 322

Picture from http://cerebro.cs.xu.edu/math/math147/02f/plimpton/plimpton322.html.

Plimpton 322

Width	Diagonal	
119	169	1
3367	4825(11521)	2
4601	6649	3
12709	18541	4
65	97	5
319	481	6
2291	3541	7
799	1249	8
481(541)	769	9
4961	8161	10
45	75	11
1679	2929	12
161(25921)	289	13
1771	3229	14
56	106(53)	15

- It consists of fifteen rows and four columns.
- Let's look at the three on the right.
- The far right is simply the numbering of the lines.

Plimpton 322

Width	Diagonal	
119	169	1
3367	4825(11521)	2
4601	6649	3
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161(25921)	289	13
1771	3229	14
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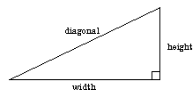
- The next two columns, with four exceptions, are the hypotenuse and one leg of integral sided right triangles.
- The four exceptions are shown with the original number in parentheses.



Lewinter & Widulski The Saga of Mathematics 55

Plimpton 322

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161(25921)	289	13
1771	3229	14
56	106(53)	15

- Line 9: 541 = $(9,1)_{60}$ and 481 = $(8,1)_{60}$.
- Line 13: 161^2 = 25921.
- Line 15: Their 53 is half the correct value.
- But line 2 has an unexplained error.

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Plimpton 322

- A *Pythagorean triple* is a set of numbers which correspond to the integral sides of a right triangle.
- For example, (3, 4, 5) and (5, 12, 13).
- Pythagorean triples can be written parametrically as $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$. (see Chapter 3)
- It seems Babylonians were aware of this.

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Plimpton 322

- The fourth column gives the values of $(c/a)^2$.
- These values are the squares of the *secant* of angle *B* in the triangle.
- This makes the tablet the oldest record of trigonometric functions.
- It is a secant table for angles between 30° and 45°.

$(119/120)^2$	119	169	1
$(3367/3456)^2$	3367	4825	2
$(4601/4800)^2$	4601	6649	3
$(12709/13500)^2$	12709	18541	4
$(65/72)^2$	65	97	5
$(319/360)^2$	319	481	6
$(2291/2700)^2$	2291	3541	7
$(799/960)^2$	799	1249	8
$(481/600)^2$	481	769	9
$(4961/6480)^2$	4961	8161	10
$(3/4)^2$	45	75	11
$(1679/2400)^2$	1679	2929	12
$(161/240)^2$	161	289	13
$(1771/2700)^2$	1771	3229	14
$(28/45)^2$	56	106	15

Lewinter & Widulski The Saga of Mathematics 58

Plimpton 322

$(119/120)^2$	119	169	1
$(3367/3456)^2$	3367	4825	2
$(4601/4800)^2$	4601	6649	3
$(12709/13500)^2$	12709	18541	4
$(65/72)^2$	65	97	5
$(319/360)^2$	319	481	6
$(2291/2700)^2$	2291	3541	7
$(799/960)^2$	799	1249	8
$(481/600)^2$	481	769	9
$(4961/6480)^2$	4961	8161	10
$(3/4)^2$	45	75	11
$(1679/2400)^2$	1679	2929	12
$(161/240)^2$	161	289	13
$(1771/2700)^2$	1771	3229	14
$(28/45)^2$	56	106	15

- What did they want with a secant table?
- Since Babylonians never introduced a measure of angles in the modern sense, it is believed that this was just a benefit of their goal in measuring areas of squares on the sides of right triangles.

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Pythagorean Problems

- *4 is the length and 5 the diagonal. What is the breadth?*
- Solution: *Its size is not known. 4 times 4 is 16. 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9. 3 is the breadth.*

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Pythagorean Problems

- *A beam of length 0;30 stands in an upright position against the wall. The upper end has slipped down a distance 0;6. How far did the lower end move from the wall?*
- Solution: *A triangle is formed with height the difference 0;24 and diagonal 0;30. Squaring 0;30 gives 0;15. Squaring 0;24 gives 0;9,36. Square root the difference 0;5,24 and the result is 0;18.*

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Why did they use 60?

- Three reasons have been suggested:
  - Theon of Alexandria believed as many other historians that 60 has many factors making certain fractions have nice sexagesimal representation.
  - "Natural" origin – the Babylonian year contained 360 days, a higher base of 360 was chosen initially then lowered to 60.
  - Merger of two people, one with a decimal and one with a base-6 system.

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