Chapter 1

Oh, So Mysterious
Egyptian Mathematics!

Primitive Man

- Hunter/gatherers
- Counted
- Simple
- Notches on wolf bone
- Groups of pebbles and stones
- Development of a simple grouping system

Early Civilizations

- Humans discovered agriculture
- Need for a calendar
- Trading or bartering of services and goods
- Production of goods
- An ability to observe the universe
- Mathematics is required

Egyptian Civilization

- Civilization reached a high point in Egypt at a very early time, 3000 B.C.
- By 3000 BC, Egypt had developed agriculture making use of the wet and dry periods of the year
- The Nile flooded during the rainy season
- Knowing when the flooding was going to arrive was extremely important
- The study of astronomy was developed to provide this calendar information

Egyptian Civilization

- Egyptian civilization required administration, a system of taxes, and armies to support it
- As the society became more complex,
  - Written records were required
  - Computations needed to be done as the people bartered their goods
- A need for counting arose, then writing and numerals were needed to record transactions

Egyptian Society

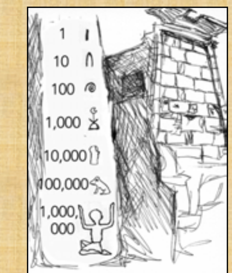
- Established a writing system for words and numerals– *hieroglyphics.*
- Kept written records – *papyrus.*
  - The Rhind/Ahmes papyrus
  - The Moscow papyrus
- Developed a calendar and watched the skies for astrological events – *astronomy.*

Egyptian Society

- Built complex structures – *pyramids, sphinx, etc.*
- For example, the Great Pyramid at Giza was built around 2650 BC and it is truly an extraordinary feat of engineering.
- All of these things required mathematics.

Egyptian Mathematics

- Simple grouping system (hieroglyphics)
- The Egyptians used the *stick* for 1, the *heel bone* for 10, the *scroll* for 100, the *lotus flower* for 1,000, the *bent finger* or *snake* for 10,000, the *burbot fish* or *tadpole* for 100,000 and the *astonished man* for 1,000,000.

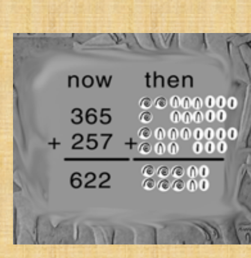


Egyptian Numerals

- Using these symbols we can write large numbers simply by grouping them appropriately
- For example, the number 243,526 would be written as:

Addition and Subtraction

- When adding, *ten* of any symbol would be replaced by *one* of the next higher symbol
- When subtracting, if you need to borrow, simply replace *one* of the next higher symbol by *ten* of the necessary symbols



Egyptian Multiplication

- Unique method which they correctly viewed as repeated addition.
- Based on doubling and is also known as *the didactic method.*
- Starting with one and doubling, they obtained a never-ending sequence of numbers: 1, 2, 4, 8, 16, 32, 64, 128, ...
- These numbers are the powers of two: 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, …

Egyptian Multiplication

- Egyptians figured out is that any integer can be written as a sum of the powers of two without repeating any of them
- For example,
  - 11 = 8 + 2 + 1
  - 23 = 16 + 4 + 2 + 1
  - 44 = 32 + 8 + 4
  - 158 = 128 + 16 + 8 + 4 + 2

Egyptian Multiplication

- Suppose we want to multiply 12 x 17.
- Start with 1 and 17.
- Keep doubling both numbers until the left side gets as close as possible to, but not larger than 12.

1	17
2	34
4	68
8	136

Egyptian Multiplication

- Subtract the left side numbers from 12 until you reach 0.
- Star the left side numbers that are being subtracted.
- In this case,
  - ✓ 12 – 8 = 4
  - ✓ 4 – 4 = 0

	1	17
	2	34
*	4	68
*	8	136

Egyptian Multiplication

- To obtain the answer, add the corresponding right side numbers of the starred positions.
- In this case,
  - ✓ 136 + 68 = 204
- So, 12 x 17 = 204.
- Neat!

	1	17
	2	34
*	4	68
*	8	136

Why it works?

- This ingenious method relies on the distributive law $a \times (b + c) = a \times b + a \times c$
- Since 12 = 4 + 8, we can write

$$17 \times 12 = 17 \times (4 + 8) = 17 \times 4 + 17 \times 8 = 68 + 136 = 204$$

- Not bad for thousands of years ago!

Egyptian Fractions

- Egyptians recognized that fractions begin with the so-called *reciprocals* of whole numbers, like 1/3 or 1/8.
- Egyptians used only fractions whose numerator was 1, like 1/3 or 1/8 (with the exception of the fraction 2/3.)
- A fraction whose numerator is one is called a *unit fraction*.

Egyptian Fractions

- Egyptians denoted unit fractions by placing an *eye* over them, e.g., to the right we see the fractions 1/10 and 1/123.
- Two exceptions existed one for 1/2 and the other for 2/3.

Egyptian Fractions

- These two fractions had their own symbols:
  - 1/2 had a sign of its own (), and
  - 2/3 had its own symbol ().
- All other fractions were written as the sum of progressively smaller unit fractions.
- It is interesting that Egyptian fractions were used well into the middle ages, in Europe.

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Egyptian fractions

- Egyptians insisted on writing fractions such as 3/4 or 7/8 as sums of **unique** unit fractions
  - 3/4 = 1/2 + 1/4
  - 7/8 = 4/8 + 2/8 + 1/8 = 1/2 + 1/4 + 1/8
- It is indeed a fact that all fractions can be written as the sum of unique unit fractions
- This fact has intrigued mathematicians for millennia.

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Unit Fractions

- There are several methods for writing a fraction as the sum of unit fractions.
  - The Egyptian method
  - Decomposition using proper divisors
  - Sylvester's method
  - The Modern method
  - The Splitting method

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The Egyptian Method

- This method consists of multiplying the denominator by unit fractions (1/2, 1/3, 1/4, 1/5, …) to obtain numbers that will add up to the numerator.
- For example, if the fraction is 5/6, we would take ½ x 6 = 3 and 1/3 x 6 = 2
- Since 3 + 2 = 5 (the numerator), $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$

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Write 7/18 Using Unit Fractions (The Egyptian Method)

Denominator = 18		
~~½~~	~~9~~	(too big)
1/3	6	(need 1 more)
1/18	1	
	7	

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Unit Fraction Rule

- If you need $\frac{1}{n}$,

  use $$\frac{1}{n \times \text{denominator}}$$

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Using Proper Divisors

- This method consists of examining the divisors of the denominator for factors that will sum to the numerator.
- For example, suppose we want to write 11/18 as the sum of unit fractions
  - The factors of 18 are 1, 2, 3, 6, 9, and 18.
  - Since 11 = 9 + 2, we can write $\frac{11}{18} = \frac{9}{18} + \frac{2}{18}$

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Using Proper Divisors

- After reducing, we have $\frac{11}{18} = \frac{9}{18} + \frac{2}{18} = \frac{1}{2} + \frac{1}{9}$
- Suppose, on the other hand, we want to write 11/15 as the sum of unit fractions
  - The factors of 15 are 1, 3, 5, and 15.
  - It appears to be impossible!
- In this case we can rename the fraction 11/15 as 22/30.

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Using Proper Divisors

- The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30.
- And, 22 = 15 + 5 + 2, so we can write

$$\frac{11}{15} = \frac{22}{30} = \frac{15}{30} + \frac{5}{30} + \frac{2}{30} = \frac{1}{2} + \frac{1}{6} + \frac{1}{15}$$

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Sylvester's Method

- Originally, developed by Fibonacci (1175-1250).
- Rediscovered by J.J. Sylvester (1814-1897) in 1880.
- Subtract from the given fraction the largest unit fraction possible.
- If the result is not a unit fraction, repeat the procedure as many times as necessary to obtain all unit fractions.

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Sylvester's Method

- Note: $\frac{a}{b} - \frac{1}{c} = \frac{ac-b}{bc}$
- Therefore, $ca > b$.
- Use the multiplier of the numerator that yields the smallest result larger than the denominator.
- Then, the multiplier becomes the denominator for the unit fraction to be subtracted.

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The Modern Method

- Similar to Sylvester's method.
- Use the multiplier of the numerator that yields the smallest result larger than the denominator.
- Then set up the equation: $(M)(N) = D + C$

  where N = numerator of the given fraction, D = denominator of the given fraction, M = multiplier that is chosen, C = constant that must be used to create the equation.

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The Modern Method

- Then, divide the equation through by $(M)(D)$.
- If this does not result in unit fractions, repeat the procedure as many times as necessary to obtain all unit fractions.

The Splitting Method

- Write the given fraction as the sum of unit fractions using repetitions.
- Then apply the formula $\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}$
- For example,

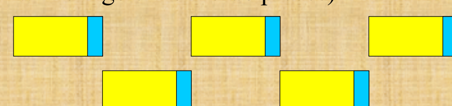
$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3}$$
$$=\frac{1}{3}+\left(\frac{1}{4}+\frac{1}{12}\right)=\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$$

Uses of Egyptian Fractions

- Egyptian fractions are useful for comparing fractions.
- Which is larger: 4/5 or 7/10?
- Writing both as sums of unit fractions
  - 4/5 = 1/2 + 1/5 + 1/10
  - 7/10 = 1/2 + 1/5
- We can now see that 4/5 is larger by exactly 1/10.

Uses of Egyptian Fractions

- Suppose Neferet and Seth want to divide 5 loaves of bread among 6 of their friends.
- Today, we would would give each person 5/6th of a loaf (5 people get the yellow piece while 1 gets the 5 blue pieces)

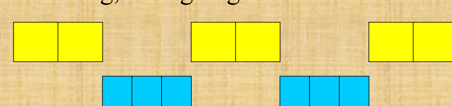


Uses of Egyptian Fractions

- This division is not very fair. Someone with the large one piece would argue that the person with 5 pieces has more (1 piece versus 5 pieces)
- The person with the 5 small pieces would argue that the people with the large piece have more (large piece versus small pieces)

Uses of Egyptian Fractions

- Using Egyptian fractions things are much more equitable.
- Writing 5/6 = 1/2+ 1/3 we can give each person one yellow and one blue piece
- Amazing, no arguing!



Egyptian Division

- Suppose we want to divide 25 by 4.
- Start with 1 and the divisor 4.
- Keep doubling both numbers until the right side gets as close as possible to, but not larger than 25.

1	4	
2	8	*
4	16	*

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Egyptian Division

- Subtract the right side numbers from 25 until you can no longer subtract.
- Star the right side numbers that are being subtracted.
  - ✓ $25 - 16 = 9$
  - ✓ $9 - 8 = 1$

1	4	
2	8	*
4	16	*

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Egyptian Division

- What is left is the *remainder*, in this case, the remainder is 1.
- To obtain the answer or *quotient*, add the corresponding left side numbers of the starred positions.
  - ✓ $4 + 2 = 6$

1	4	
2	8	*
4	16	*

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Egyptian Division

- Thus, $25 \div 4 = 6\ R\ 1$.
- The Egyptians would have used unit fractions to write the answer, so for them

$$25 \div 4 = 6\frac{1}{4} = 6 + \frac{1}{4}$$

1	4	
2	8	*
4	16	*

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Egyptian Geometry

- One reason the ancient Egyptians had to deal with multiplication involved geometry and measurement.
- Measurement involves questions like "how much", "how big", "how fast", and "how heavy".
- The mathematician then must conjure up a "unit" which translates the above questions into "how many *cupfuls*", "how many *inches*", "how many *miles per hour*", and "how many *pounds*".

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Egyptian Geometry

- The Egyptians took an enormously giant step by inventing a unit of area from a unit of length by forming a square unit of area!
- Let's use square feet for simplicity.
- A foot is a unit of length — but a tile of length and width one foot, i.e., a unit square tile, can be said to have area one (one square foot, that is).

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Egyptian Geometry

- A rectangular room of length 30 feet and width 20 feet, can be tiled with 30 rows of twenty tiles each.
- Instead of repeatedly adding twenty thirty times, we have that 30 x 20 = 600, and the room has a floor area of 600 square feet.
- “How much area” became “how many square feet” and this is how we measure area today!

Egyptian Geometry

- The Egyptians had many of the formulas for area and volume that we have today
  - The area of a rectangle $A = LW$.
  - The area of a triangle $A = \frac{1}{2} BH$.
  - The volume of a rectangular solid $V = LWH$.
  - The volume of a pyramid $V = (1/3)\times HB^2$.
  - The volume of a frustum or truncated pyramid $V = (1/3)\times H(B_1^2 + B_1 B_2 + B_2^2)$

Area of the Circle

- The Rhind/Ahmes Papyrus Problem #50 states: A circular field has diameter 9 khet. What is its area? [Note: 1 *khet* is 100 cubits, and 1 meter is about 2 cubits. A *setat* is a measurement of area equal to what we would call a square khet.]
- The solution says, “Take from its diameter one ninth part. The result shall form the side of a square whose area is equal to that of the circle.”

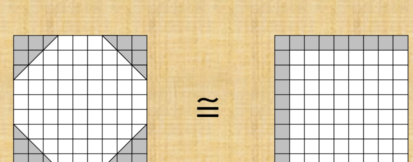
Area of the Circle

- Following this, we subtract 1/9 of the diameter which leaves 8 khet. The area of the square (hence, the circle) is 8 times 8, or 64 setat.
- The Egyptians were using a formula for the area of a circle as $A = (8d/9)^2 = 64d^2/81$
- Today we know the area of a circle of diameter d is $A = \pi(d/2)^2 = \pi\, d^2/4$.

Egyptian Value of Pi

- Let’s assume $64 = \pi\, 9^2/4 = \pi\, 81/4$, then $\pi = 256/81 = 3 + 1/9 + 1/27 + 1/81 \cong 3.1605$.
- While 256/81 can be written in infinitely many ways using unit fractions, the Egyptians preferred 3 + 1/9 + 1/27 + 1/81 to say 3 + 1/13 + 1/17 + 1/160 since the former uses only powers of 3 in the denominator!

How They Did It?



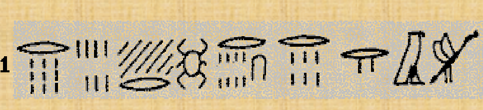
The Moscow Papyrus

- The Moscow Papyrus (~1850 B.C.) contains 25 problems and solutions.
- The author is unknown.
- There are many Internet sites dedicated to this piece of mathematical history.
- Search at Google!

The Moscow Papyrus

- The translation of line 11 from problem #10 is

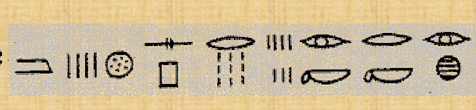
"After subtracting 2/3 + 1/6 + 1/18. You get 7 + 1/9."

11 

The Moscow Papyrus

- The translation of line 12 from problem #10 is

"Multiply 7 + 1/9 by 4 + ½."

12 

The Moscow Papyrus

- **Problem 14.** Volume of a frustum. The scribe directs one to square the numbers two and four and to add to the sum of these squares the product of two and four. Multiply this by one third of six. "See, it is 56; your have found it correctly."

The Moscow Papyrus

- Moscow #6: We are given a rectangular enclosure of area 12 setat. The width is ¾ of the length. Find both the length and the width.
- Moscow #7: The height of a triangle is 2 and ½ times the base. The area is 20. Find the base and the height.
- Moscow #17: The height of a triangle is 2/5 of the base. The area is 20. Find the base and the height.

The Rhind/Ahmes Papyrus

- The Rhind papyrus is named after the Scottish Egyptologist A. Henry Rhind, who purchased it in Luxor in 1858.
- It was written around 1650 BC by the scribe Ahmes who claims that he is copying a document that is 200 years older.
- It claims to be a ``thorough study of all things, insight into all that exists, and knowledge of all obscure secrets."

The Rhind/Ahmes Papyrus

- The Rhind/Ahmes Papyrus contains 85 problems and solutions.
- Problems 41-43, 48, and 50 of the Rhind/Ahmes Papyrus deal with finding the area of a circle.

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The Rhind/Ahmes Papyrus

- Rhind #41: Find the volume of a cylindrical granary of diameter 9 and height 10.
- Rhind #43: A cylindrical granary has a diameter 9 and height 6. What is the amount of grain that goes into it?
- Rhind #48: Compare the areas of a circle of diameter 9 and its circumscribing square.
- Rhind #51: What is the area of a triangle of side 10 and base 4?

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Egyptian Astronomy

- Egyptians eventually noticed the *periodic* (repetitive) behavior of the *trajectories* (paths) of heavenly bodies and, of course, the regular progression of night and day.

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Egyptian Astronomy

- As well as the sun's daily routine — rising in the east and setting in the west.
- The equal time intervals between 'new moons', approximately 28 days, afforded the ancient civilizations a means of time measurement.
- This is still the basis of some calendars today.

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Egyptian Astronomy

- The Egyptians needed a calendar for various reasons, like
  1. knowing when to plant and harvest crops,
  2. predicting the annual flooding of the Nile River, and
  3. recording important events, like the Pharaoh's birthday.
- Note that the flooding of the Nile was tied to the helical rising of Sirius and not the calendar since it did not remain in synch with sun.

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Egyptian Astronomy

- Having a calendar involves observing the shift on the horizon of the rising of the sun and several prominent stars and planets.
- The eye sweeps out a huge circle as it beholds the entire horizon.
- Egyptian geometry was not confined to land surveying and architecture. It played an integral part in locating planets in the sky.

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Egyptian Astronomy

- Points on the horizon can be measured by the angle between the observer's line of sight and a fixed line.
- The ancients imagined the sky is an enormous hemisphere and the earth is a flat disc sharing a common circular boundary with the sky.

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Egyptian Astronomy

- An observer looking for the North Star at 1:00 am would use two angles.
- The first locates a point on the common boundary of the disc and hemisphere, i.e., the horizon, and the second is the star's 'angle of elevation'.

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Egyptian Calendar

- The Egyptians followed a calendar system of 360 days, with three seasons, each made up of 4 months, with thirty days in each month, plus five additional days known as "the yearly five days".
- The seasons of the Egyptians corresponded with the cycles of the Nile.
- The beginning of the year, also called "the opening of the year", was marked by the emergence of the star Sirius.
- The additional five days, were times of great feasting and celebration for the Egyptians.

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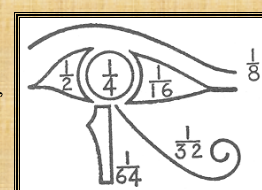
Eye of Horus

- The Eye of Horus, sometimes called the '*oudjat*' was a talisman symbolizing the wholeness of the body, physical health, clear vision, abundance and fertility.



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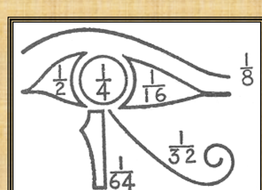
Eye of Horus

- In time the different sections of the eye came to represent fractions, specifically for measures of grains and liquids.



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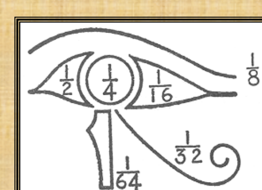
Eye of Horus

- The total of the '*oudjat*' is 63/64.
- The missing 1/64 would be made up by '*Thot*' (the God of the scribes) to any scribe who sought and accepted his protection.



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Additional Topics

- The Egyptian hieratic numerals is an example of a ciphered system. (Exercise 21)
- The Egyptian method of false position shows their ability to solve linear equations. (Exercise 19 and 20)
- Ideas for Papers